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Some Co-Birkhoff-Type Theorems

Jesse Hughes

Department of Computer Science
University of Nijmegen
Toernooiveld 1
6525 ED
Nijmegen, The Netherlands
jesseh@cs.kun.nl

Abstract

We consider the dual of Theorem 1 from [33], relating closure conditions on subcategories with projectivity classes for collections of discrete cocones. We extend these results by adding a new operator for closure under domains of epis and show that this corresponds to taking projectivity classes for cocones with vertex a subobject of the terminal object. We show that these theorems apply to categories of coalgebras under reasonable assumptions on the base category and endofunctor. Lastly, we discuss cofree for **V** coalgebras in this setting and give examples of so-called Horn covarieties of coalgebras.

1 Introduction

There has been much work in recent years dualizing the Birkhoff variety theorem to yield a covariety theorem for coalgebras. This goal was first raised in [24] and an early result appeared in [36] for coalgebras over **Set**. Since then, the co-Birkhoff theorem has been further developed in [21,20,19] for coalgebras over **Set**, and also in [26,27,22,23,4,29] for coalgebras in a more general setting. (Also, a coalgebraic analogue to the variety theorem has been explored in [18], but this is an analogue as opposed to a formal dual.) Although the details may differ somewhat (notably on what a coequation is and what coequation satisfaction means), the basic covariety theorem is recognizable: A collection **V** of coalgebras is closed under images, coproducts and subobjects iff **V** is coequationally definable (whatever that may mean).

In addition, quasi-covariety theorems were explored in many of these articles, notably [20,19,26,27,23]. These theorems state an equivalence between closure of classes of coalgebras under images and coproducts (but not necessarily subobjects) and definability via implications between coequations (described as "modal rules" in [26,27]).

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A categorical analysis of the corresponding algebraic theorems goes back to [12]. There, we find an early presentation of Birkhoff's variety theorem in a categorical setting. In the classical setting, a set S of equations over a set X of variables is a relation on UFX, the carrier of the free algebra over X, as in

$$S \overset{s_1}{\underset{s_2}{\rightrightarrows}} UFX.$$

An algebra $\mathcal{A}$ satisfies S just in case every algebra homomorphism $FX \to \mathcal{A}$ equalizes s_1 and s_2. Equivalently, we may consider the quotient $\mathcal{Q}$ of the adjoint transposes of s_1 and s_2, as in

$$FS \rightrightarrows FX \twoheadrightarrow \mathcal{Q}.$$

Clearly, $\mathcal{A} \models S$ just in case every homomorphism $FX \to \mathcal{A}$ factors through $\mathcal{Q}$, as shown below.

$$\begin{tikzcd} FS \arrow[r, shift left] \arrow[r, shift right] & FX \arrow[r, two heads] \arrow[d] & \mathcal{Q} \arrow[dl, dotted] \\ & \mathcal{A} & \end{tikzcd}$$

In categorical terms, then, equation satisfaction is given by injectivity with respect to some (regular epi) arrow out of a free algebra. This is a central feature of the approach found in ibid.

The "co-Birkhoff" theorems and their proofs in [27,23] mimic (but dualize) the Banaschewski and Herrlich approach. The basic notion behind "coequation satisfaction" in each thesis can be stated in terms of projectivity with respect to some regular mono into a cofree coalgebra, i.e., coequation satisfaction is the formal dual of satisfaction of sets of equations.

Note that we refer here to the dual of Birkhoff's variety theorem and related theorems, not to the dual of completeness. See [22] for a discussion of the formal dual of equational completeness.

In a series of articles, [6–9,33,32,10], H. Andréka and I. Németi continued development of injectivity as a kind of generalized equational definability. The strength of the approach is particularly striking in [33], in which Németi and I. Sain give a tightly organized presentation of a number of "Birkhoff-type" theorems. For our purposes, a Birkhoff-type theorem for a composite $\boldsymbol{X}$ of closure operators says: Given a class $\mathbf{V}$ (typically, of algebras), there is a collection S of cones such that an object $\mathcal{A}$ is in $\boldsymbol{X}\mathbf{V}$ iff $\mathcal{A}$ is S-injective. See below for a definition of *S-injective.*

We take this as our starting point. In Section 2, we dualize Németi and Sain's terminology and present their Theorem 1 (in a simpler, more restrictive setting), although we omit the proofs. By duality, a co-Birkhoff-type theorem for a composite $\boldsymbol{X}$ of closure operators is a theorem: For every class $\mathbf{V}$, there is a collection S of cocones such that for every $\mathcal{A}$, we have $\mathcal{A} \in \boldsymbol{X}\mathbf{V}$ iff $\mathcal{A}$ is S-projective. (Note: the dual of a closure operator [1] $\mathfrak{P}(\mathsf{Ob}\,\mathbb{C}) \to \mathfrak{P}(\mathsf{Ob}\,\mathbb{C})$ is

[1] Throughout, we let $\mathfrak{P}$ denote the operator taking a class to its collection of subclasses.

another closure operator on $\mathfrak{P}(\mathsf{Ob}\,\mathbb{C}^{\mathsf{op}})$, *not* an interior operator.) We omit the development of *finitary* varieties found in ibid, as this work does not dualize easily (we want to avoid discussing the dual of locally presentable categories). What we present here, then, settles only some of the issues surrounding "co-Birkhoff-type" theorems; namely we give the "infinitary" theorems. We do not investigate the notion of finitely axiomatizable classes of coalgebras.

We go beyond mere dualities in Section 3.1, by incorporating an operator $\mathcal{H}^-$ not present in [33]. This operator (closure under domains of $\mathcal{H}$-morphisms) appears in one guise or another in [21,22,28,34,35]. We show here that the operator fits in well with the other standard operators. We get six new co-Birkhoff-type theorems involving $\mathcal{H}^-$, the same number as we prove by duality in Section 2. We also show the relationship between $\mathcal{H}^-$ and two other operators, including closure under total relations, and we close with a short discussion of the corresponding dual Birkhoff-type theorems involving $\mathcal{H}^-$. These Birkhoff-type theorems appear to be new.

In Section 4, we show that, under weak assumptions on $\mathbb{C}$ and Γ, the abstract co-Birkhoff-type theorems from the preceding sections also apply to the category $\mathbb{C}_\Gamma$ of Γ-coalgebras. We follow this with a discussion of *cofree for* $\boldsymbol{X}\mathbf{V}$ coalgebras for certain compositions $\boldsymbol{X}$ of closure operators and close with a few examples of so-called Horn covarieties of coalgebras, i.e., classes of coalgebras closed under images and non-empty coproducts.

Steve Awodey, Bart Jacobs and Alexander Kurz gave advice on the presentation found here. I also appreciate the anonymous reviewer for [22], who referred to the work of Andréka and Németi. Lastly, but importantly, thanks to Hajnal Andréka for providing copies of the relevant articles.

2 The Nemeti-Sain Theorems

We consider Theorem 1 from [33] in a somewhat more restrictive setting than they use. In particular, they consider a subtle loosening of the assumption that $\mathbb{C}$ has a factorization system, whereas we do not. At present, our examples all involve factorization systems, and so allow ourselves the luxury of stronger assumptions on $\mathbb{C}$.

Let $\mathbb{C}$ have coproducts and let $\langle \mathcal{H}, \mathcal{S}\rangle$ be a *factorization system* for $\mathbb{C}$ such that $\mathbb{C}$ is $\mathcal{S}$*-well-powered*. That is, let $\mathcal{H}$ and $\mathcal{S}$ be collections[2] of arrows in $\mathbb{C}$ such that:

- $\mathsf{Isom} \subseteq \mathcal{H} \cap \mathcal{S}$ (the first of many abuses of set notation for classes);
- $\mathcal{H}$ and $\mathcal{S}$ are closed under composition;
- $\mathcal{H}$ and $\mathcal{S}$ satisfy the *diagonal fill-in property*, namely, for every commutative

[2] When we write "collection", we allow proper classes.

square

$$\begin{tikzcd} \bullet \arrow[r, two heads, "e"] \arrow[d] & \bullet \arrow[d] \arrow[dl, dotted, "f"] \\ \bullet \arrow[r, tail, "m"'] & \bullet \end{tikzcd}$$

where $e \in \mathcal{H}$ and $m \in \mathcal{S}$, there is a unique arrow f, as shown, making each triangle commute;

- every arrow f factors as $f = m \circ e$, where $e \in \mathcal{H}$ and $m \in \mathcal{S}$;
- ($\mathcal{S}$-well-poweredness) for each $C \in \mathbb{C}$, there is a set $\mathsf{Sub}(C) \subseteq \mathcal{S}$ such that, for all $m : \bullet \rightarrowtail C$ in $\mathcal{S}$, there is an $m' \in \mathsf{Sub}(C)$ and an isomorphism k such that $m = m' \circ k$.

Here and hereafter, we use $\rightarrowtail$ to denote arrows in $\mathcal{S}$ and $\twoheadrightarrow$ to denote arrows in $\mathcal{H}$. Note that, since $\mathcal{S}$ need not be a collection of monos, the notation $\mathsf{Sub}(A)$ is merely suggestive. The set $\mathsf{Sub}(A)$ comes with an evident order $\leq$, namely, if $P, Q \in \mathsf{Sub}(A)$, we say $P \leq Q$ just in case the $\mathcal{S}$-morphism $P \rightarrowtail A$ factors through the $\mathcal{S}$-morphism $Q \rightarrowtail A$.

Let $f : A \to B$ be an arrow in $\mathbb{C}$ and $f = m \circ e$ be its $\langle \mathcal{H}, \mathcal{S} \rangle$ factorization. We denote $\mathsf{dom}(m)$ ($= \mathsf{cod}(e)$) by $\mathsf{Im}(f)$ (read "image of f"). Note that Im arises as the left adjoint to the inclusion $\mathsf{Sub}(B) \hookrightarrow \mathbb{C}/B$.

Observe that, since $\mathbb{C}$ has coproducts, then $\mathsf{Sub}(A)$ has joins. Indeed, if $\{f_i : B_i \rightarrowtail A\}_{i \in I} \subseteq \mathsf{Sub}(A)$, then

$$\bigvee_{i \in I} f_i = \mathsf{Im}\big(\textstyle\sum_{i \in I} B_i \xrightarrow{[f_i]} A\big).$$

2.1 A cornucopia of closure operators

Given a poset $\langle P, \leq \rangle$, a closure operator $\mathcal{X}$ is a monad $\mathcal{X} : \langle P, \leq \rangle \to \langle P, \leq \rangle$, that is, a monotone, inflationary (sometimes called "extensive"), idempotent map $P \to P$. We define a number of closure operators, $\mathfrak{P}(\mathsf{Ob}\,\mathbb{C}) \to \mathfrak{P}(\mathsf{Ob}\,\mathbb{C})$ by

$$\begin{aligned} \mathcal{S}(\mathbf{V}) &= \{B \in \mathbb{C} \mid \exists A \in \mathbf{V}\, \exists B \rightarrowtail A\}, \\ \mathcal{H}(\mathbf{V}) &= \{B \in \mathbb{C} \mid \exists A \in \mathbf{V}\, \exists A \twoheadrightarrow B\}, \\ \mathcal{H}^{-}(\mathbf{V}) &= \{B \in \mathbb{C} \mid \exists A \in \mathbf{V}\, \exists B \twoheadrightarrow A\}, \\ \Sigma(\mathbf{V}) &= \{\sum_{i \in I} A_i \mid A_i \in \mathbf{V}\}, \\ \Sigma^{+}(\mathbf{V}) &= \{\sum_{i \in I} A_i \mid A_i \in \mathbf{V} \text{ and } I \neq \emptyset\}. \end{aligned}$$

We have overloaded the symbols $\mathcal{S}$ and $\mathcal{H}$, referring to both certain collections of arrows and the closure under (co)domains of those collections.

We write $\mathcal{X} \leq \mathcal{Y}$ just in case $\mathcal{X}\mathbf{V} \subseteq \mathcal{Y}\mathbf{V}$ for all $\mathbf{V} \subseteq \mathbb{C}$.

If $\mathcal{X}$ is a closure operator, we say that $\mathbf{V}$ is $\mathcal{X}$-closed just in case $\mathbf{V} = \mathcal{X}\mathbf{V}$.

Remark 2.1 Compositions of closure operators are not necessarily closure operators themselves, since they need not be idempotent. Given two closure operators $\mathcal{X}$, $\mathcal{Y}$, for the composition $\mathcal{X}\mathcal{Y}$ to be a closure operator, it is necessary and sufficient that $\mathcal{Y}\mathcal{X} \leq \mathcal{X}\mathcal{Y}$.

Note that if $\mathcal{X}$, $\mathcal{Y}$ and $X\mathcal{Y}$ are closure operators, and $\mathbf{W} \subseteq \mathsf{Ob}\,\mathbb{C}$, then $\mathbf{W}$ is $\mathcal{X}\mathcal{Y}$-closed just in case $\mathbf{W}$ is $\mathcal{X}$-closed and $\mathcal{Y}$-closed.

See [31] for an extended discussion of compositions of $\mathcal{S}$, $\mathcal{H}$ and Σ in categories of coalgebras over **Set**.

We will largely ignore the $\mathcal{H}^-$ operator for now, and include it here only to ensure that the lemmas which follow are stated with appropriate generality. See Section 3.1 for discussion of this operator (and the corresponding $M_{\mathcal{H}^-}$ and $K_{\mathcal{H}^-}$ operators, defined below).

We are interested in compositions of $\mathcal{H}^-$, $\mathcal{S}$, $\mathcal{H}$, Σ and Σ^+ in which the operators appear in that order. Further, we restrict to those compositions in which $\mathcal{H}$ appears. We use $\boldsymbol{X}$ to range over such compositions. Consequently, when we write $\boldsymbol{X}$, we intend that $\boldsymbol{X}$ is one of the compositions in the following table (unless stated otherwise).

$\mathcal{H}$	$\mathcal{H}\Sigma^+$	$\mathcal{H}\Sigma$	$\mathcal{S}\mathcal{H}$	$\mathcal{S}\mathcal{H}\Sigma^+$	$\mathcal{S}\mathcal{H}\Sigma$
$\mathcal{H}^-\mathcal{H}$	$\mathcal{H}^-\mathcal{H}\Sigma^+$	$\mathcal{H}^-\mathcal{H}\Sigma$	$\mathcal{H}^-\mathcal{S}\mathcal{H}$	$\mathcal{H}^-\mathcal{S}\mathcal{H}\Sigma^+$	$\mathcal{H}^-\mathcal{S}\mathcal{H}\Sigma$

Note that we ignore the redundant compositions $\mathcal{H}\Sigma\Sigma^+$, etc. Also note that $\mathcal{S}\mathcal{H}\Sigma^+$ is not the same as $\mathcal{S}\mathcal{H}\Sigma$, as evidenced by the fact that the empty class is $\mathcal{S}\mathcal{H}\Sigma^+$-closed.

We restrict to the compositions above as they arise via duality from the compositions considered in [33, Theorem 1]. We leave open the question whether some other compositions are definable by projectivity conditions.

2.2 Satisfaction as projectivity.

Let $A \in \mathbb{C}$ and $f : B \to C$ be given. We write $A \models f$ if A is f*-projective*, i.e., if for every $g : A \to C$, there is a (not necessarily unique) $h : A \to B$ such that $f \circ h = g$. More generally, let $c = \langle C, \{f_i : B_i \to C\}_{i \in I}\rangle$ be a cocone on some diagram $\{B_i\}_{i \in I}$ for an arbitrary set I. We call such a cocone *discrete*, since it is a cocone over a diagram on a small, discrete category (that is, a set). We say $A \models c$ if, for every $g : A \to C$, there is an $i \in I$ and a map $h : A \to B_i$ such that $g = f_i \circ h$, that is, if A is projective with respect to the cocone c. In particular, if $I = \emptyset$, then $A \models c$ just in case there is no map $A \to C$. If T is a collection of cocones, we write $A \models T$ (or say that A is T-projective) just in case $A \models c$ for every $c \in T$. We let $\mathbf{Proj}(T)$ (read *the projectivity class for* T) denote the full subcategory of $\mathbb{C}$ consisting of all those objects A such that $A \models T$.

Let $\mathsf{Cocone}(\mathbb{C})$ denote the collection of discrete cocones of $\mathbb{C}$. That is, the objects of $\mathsf{Cocone}(\mathbb{C})$ are pairs $\langle C, \{f_i : C_i \to C\}_{i \in I}\rangle$. Let $\mathsf{Cocone}(\mathbb{C}, A)$ be the subcategory of $\mathsf{Cocone}(\mathbb{C})$ consisting of those cocones rooted at $A \in \mathbb{C}$. Evidently, **Proj** is a contravariant functor (i.e., order-reversing operator) $\mathfrak{P}(\mathsf{Cocone}(\mathbb{C})) \to \mathfrak{P}(\mathsf{Ob}(\mathbb{C}))$.

The formal dual to T-projectivity is T-injectivity. In particular, an object A in $\mathbb{C}$ is $\mathcal{S}$-injective just in case, for every $f : C \twoheadrightarrow B$ in $\mathcal{S}$ and every $g : C \to A$, there is an $h : B \to A$ (an "extension" of g) such that $h \circ f = g$. As usual, one says that $\mathbb{C}$ *has enough $\mathcal{S}$-injectives* just in case for every $C \in \mathbb{C}$, there is an $\mathcal{S}$-morphism $C \twoheadrightarrow A$ for some $\mathcal{S}$-injective A.

We define the following subclasses of $\mathsf{Cocone}(\mathbb{C})$. (For the definition of $M_{\mathcal{H}^-}$, we assume that $\mathbb{C}$ has a terminal object 1.)

$$\begin{aligned}
M_{\mathcal{S}} &= \{\langle C, \{f_i : B_i \longrightarrow C\}_{i \in I}\rangle \mid C \text{ is } \mathcal{S}\text{-injective}\} \\
M_{\mathcal{H}} &= \{\langle C, \{f_i : B_i \rightarrowtail C\}_{i \in I}\rangle \mid \forall i \in I \,.\, f_i \in \mathcal{S}\} \\
M_{\mathcal{H}^-} &= \{\langle C, \{f_i : B_i \longrightarrow C\}_{i \in I}\rangle \mid C \in \mathsf{Sub}(1)\} \\
M_{\Sigma} &= \{\langle C, \{f_i : B_i \longrightarrow C\}_{i \in I}\rangle \mid \mathbf{card}(I) = 1\} \\
M_{\Sigma^+} &= \{\langle C, \{f_i : B_i \longrightarrow C\}_{i \in I}\rangle \mid \mathbf{card}(I) \leq 1\}
\end{aligned}$$

If $\boldsymbol{X} = \mathcal{X}_1 \ldots \mathcal{X}_n$, we define $M_{\boldsymbol{X}} = M_{\mathcal{X}_1} \cap \ldots \cap M_{\mathcal{X}_n}$.

We next define a family $K_{\boldsymbol{X}}$ of contravariant operators taking collections $\mathbf{V}$ of objects of $\mathbb{C}$ to collections of cocones of $\mathbb{C}$. More precisely, for each $\boldsymbol{X}$, we will define

$$K_{\boldsymbol{X}} : \mathfrak{P}(\mathsf{Ob}\,\mathbb{C}) \longrightarrow \mathfrak{P}(M_{\boldsymbol{X}}).$$

Given a full subcategory $\mathbf{V}$ of $\mathbb{C}$, for each list $\boldsymbol{X}$, as above, we define

$$K_{\boldsymbol{X}}\mathbf{V} = \{c \in M_{\boldsymbol{X}} \mid \mathbf{V} \models c\}.$$

The following theorem shows that a collection $\mathbf{V}$ is the projectivity class for some collection T of cocones (of a particular "kind" $\boldsymbol{X}$) just in case $\mathbf{V}$ is the projectivity class for $K_{\boldsymbol{X}}\mathbf{V}$.

Theorem 2.2 *Let $\mathbf{V}$ be a full subcategory of $\mathbb{C}$ and let $\boldsymbol{X}$ be a list of the operators above. Then, there is a collection $T \subseteq M_{\boldsymbol{X}}$ such that $\mathbf{V} = \mathbf{Proj}(T)$ just in case $\mathbf{V} = \mathbf{Proj}(K_{\boldsymbol{X}}\mathbf{V})$.*

Proof. If $\mathbf{V} = \mathbf{Proj}(K_{\boldsymbol{X}}\mathbf{V})$, then the theorem trivially follows, since $K_{\boldsymbol{X}}\mathbf{V} \subseteq M_{\boldsymbol{X}}$. Conversely, suppose $\mathbf{V} = \mathbf{Proj}(T)$ where $T \subseteq M_{\boldsymbol{X}}$. Then, since $T \subseteq K_{\boldsymbol{X}}\mathbf{V}$,

$$\mathbf{Proj}(K_{\boldsymbol{X}}\mathbf{V}) \subseteq \mathbf{Proj}(T) = \mathbf{V}.$$

□

2.3 The co-Birkhoff-type theorems

We omit the proof of the following theorem, as it is the straightforward dual of [33, Theorem 1]. The corollary which follows can also be found there.

Theorem 2.3 *Let $\mathbb{C}$ have all coproducts, a factorization system $\langle\mathcal{H}, \mathcal{S}\rangle$ with $\mathcal{S} \subseteq$ Mono and enough $\mathcal{S}$-injectives and, further, that $\mathbb{C}$ is $\mathcal{S}$-well-powered. Then for any $\boldsymbol{X}$ in*

$$\{\mathcal{H}, \mathcal{H}\Sigma^+, \mathcal{H}\Sigma, \mathcal{S}\mathcal{H}, \mathcal{S}\mathcal{H}\Sigma^+, \mathcal{S}\mathcal{H}\Sigma\},$$

we have $\mathbf{Proj}(K_{\boldsymbol{X}}\mathbf{V}) = \boldsymbol{X}\mathbf{V}$.

Remark 2.4 The assumptions on Theorem 2.3 can be weakened in particular cases. The assumption that $\mathbb{C}$ has all (non-empty, resp.) coproducts is used only for $\boldsymbol{X} \geq \Sigma$ ($\boldsymbol{X} \geq \Sigma^+$, resp.). Enough $\mathcal{S}$-injectives is used for those composites $\boldsymbol{X} \geq \mathcal{S}$ and $\mathcal{S} \subseteq$ Mono for those composites not involving $\mathcal{S}$.

Corollary 2.5 *Let*

$$\boldsymbol{X} \in \{\mathcal{H}, \mathcal{H}\Sigma^+, \mathcal{H}\Sigma, \mathcal{S}\mathcal{H}, \mathcal{S}\mathcal{H}\Sigma^+, \mathcal{S}\mathcal{H}\Sigma\}.$$

Then $\boldsymbol{X}$ is a closure operator.

Remark 2.6 Németi and Sain prove somewhat more general results in [33]. In particular, they consider an additional operator $\mathcal{Z}$ as well, which does not appear in what follows. It is worthwhile to ask whether this operator leads to any interesting classes of coalgebras. This question is left for future research.

Example 2.7 Consider the category **Set** and the full subcategory $\mathbf{Set}_{\mathsf{fin}}$ of finite sets. It is easy to see that $\mathbf{Set}_{\mathsf{fin}} = \mathcal{S}\mathcal{H}\mathbf{Set}_{\mathsf{fin}}$, where $\mathcal{H}$ = Epi and $\mathcal{S}$ = Mono. One can show that $\mathbf{Set}_{\mathsf{fin}}$ is the projectivity class of the following cocone. Take the cocone with vertex $\mathbb{N}$ and with arrows all of the monic morphisms $n \rightarrowtail \mathbb{N}$. Note that we take *all* of such monic morphisms and not just the inclusions $n \hookrightarrow \mathbb{N}$. It is easy to see that $\mathbf{Set}_{\mathsf{fin}}$ is the projectivity class of the resulting cocone.

Remark 2.8 Since **Set** is regular (that is, since the evident factorization system for **Set** is *stable*), one sees that $\mathcal{H}\mathcal{S}\mathbf{Set}_{\mathsf{fin}} = \mathcal{S}\mathcal{H}\mathbf{Set}_{\mathsf{fin}}$ and hence $\mathbf{Set}_{\mathsf{fin}}$ can also be described as the injectivity class of a particular cone. The description of the cone is left to the interested reader.

3 Behavioral classes

We now extend the previous dualized results of Németi and Sain to consider the $\mathcal{H}^-$ operator. This operator and its corresponding cocone operators arise naturally in a coalgebraic setting, where $\mathcal{H}^-\mathcal{H}\mathbf{V}$ closes $\mathbf{V}$ under total bisimulations. We start by proving the Birkhoff-type theorems involving the $\mathcal{H}^-$ operator. In Section 3.2, we examine two related operators to help motivate the definition found here.

Throughout this section, we add an extra assumption regarding our factorization system $\langle\mathcal{H}, \mathcal{S}\rangle$. Namely, we assume that SplitEpi $\subseteq \mathcal{H}$. Note that if $\mathbb{C}$ has binary products, then the assumption that $\mathcal{S} \subseteq$ Mono entails that $\mathcal{H} \subseteq$ Epi and so SplitEpi $\subseteq \mathcal{H}$ (see [2, Theorem 14.11]).

We make explicit use of this assumption in Lemma 3.2.

$\mathcal{H}^-$-closed collections of coalgebras have previously been discussed in [34], [35], [21], [18] and [23].

3.1 $\mathcal{H}^-$ co-Birkhoff-type theorems

Throughout this section, we assume that $\mathbb{C}$ satisfies the following conditions.

(i) $\mathbb{C}$ has all coproducts;

(ii) $\mathbb{C}$ has a factorization system $\langle \mathcal{H}, \mathcal{S} \rangle$;

(iii) $\mathcal{S} \subseteq$ Mono;

(iv) $\mathbb{C}$ has enough $\mathcal{S}$-injectives;

(v) $\mathbb{C}$ is $\mathcal{S}$-well-powered;

(vi) SplitEpi $\subseteq \mathcal{H}$;

(vii) $\mathbb{C}$ has a terminal object, 1.

of Theorem 2.3. This next lemma is the "easy" direction of the equation $\mathcal{H}^- \boldsymbol{X}\mathbf{V} = \mathbf{Proj}(K_{\mathcal{H}^-\boldsymbol{X}}\mathbf{V})$.

Lemma 3.1 *Let* $\mathbf{V}$ *be a full subcategory of* $\mathbb{C}$*.*

$$\mathcal{H}^- \boldsymbol{X}\mathbf{V} \subseteq \mathbf{Proj}(K_{\mathcal{H}^-\boldsymbol{X}}\mathbf{V}).$$

Proof. Let $A \twoheadrightarrow B$ be given, $B \in \boldsymbol{X}\mathbf{V}$. Then $B \in \mathbf{Proj}(K_{\boldsymbol{X}}\mathbf{V})$. Let $c = \langle C, \{f_i : C_i \rightarrowtail C\}_{i \in I} \rangle$ be a cocone in $K_{\mathcal{H}^-\boldsymbol{X}}\mathbf{V}$, so that $C \leq 1$, and let $f : A \to C$ be given, as shown below.

$$\begin{tikzcd}
C_i \arrow[r, tail] & C \arrow[d, tail] & A \arrow[l, "f"'] \arrow[d, two heads] \\
 & 1 & B \arrow[l] \arrow[ul, dotted] \arrow[ull, dotted]
\end{tikzcd}$$

Then, by the diagonal fill-in principle, there is a morphism $B \to C$, as shown, making each triangle in the square commute. Hence, there is a map $B \to C_i$, for some $i \in I$, as shown, making the corresponding triangle commute and so f factors through C_i via the composite $A \twoheadrightarrow B \longrightarrow C_i$. □

Before continuing to prove the "hard" inclusions, we define an operator, $\mathcal{F}^{\mathbf{V}} : \mathbb{C} \to \mathfrak{P}(\mathbb{C})$, by

$$\mathcal{F}^{\mathbf{V}} A = \{\mathsf{Im}(f : B \longrightarrow A) \mid B \in \mathbf{V}\}.$$

Note that for any $\mathbf{V}$, A, we have $\mathbf{V} \models \langle A, \mathcal{F}^{\mathbf{V}} A \rangle$.

Lemma 3.2 *Let* $\mathbf{V}$ *be a full subcategory of* $\mathbb{C}$.

$$\mathbf{Proj}(K_{\mathcal{H}^{-}\mathcal{H}}\mathbf{V}) = \mathcal{H}^{-}\mathcal{H}\mathbf{V}$$

Proof. $\mathbf{Proj}(K_{\mathcal{H}^{-}\mathcal{H}}\mathbf{V}) \supseteq \mathcal{H}^{-}\mathcal{H}\mathbf{V}$ by Lemma 3.1.

$$\begin{tikzcd} B \arrow[d] & A \arrow[dl, dotted] \arrow[d] \arrow[dr, "!_A"] & \\ \mathsf{Im}\, f \arrow[r, tail] & C \arrow[r, tail] & 1 \end{tikzcd}$$

To prove that $\mathbf{Proj}(K_{\mathcal{H}^{-}\mathcal{H}}\mathbf{V}) \subseteq \mathcal{H}^{-}\mathcal{H}\mathbf{V}$, suppose that $A \models K_{\mathcal{H}^{-}\mathcal{H}}\mathbf{V}$, and let $C = \mathsf{Im}(!_A)$, as shown above. Let $c = \langle C, \mathcal{F}^{\mathbf{V}}C\rangle$. Clearly $\mathbf{V} \models c$ and so $A \models c$ as well. Thus, there is some $B \in \mathbf{V}$, $f : B \to C$ such that $!_A$ factors through $\mathsf{Im}(f)$, as shown.

Applying the diagonal fill-in principle to the square

$$\begin{tikzcd} A \arrow[r, two heads] \arrow[d] & C \arrow[dl, dotted] \arrow[d, equal] \\ \mathsf{Im}(f) \arrow[r, tail] & C \end{tikzcd}$$

we see that $\mathsf{Im}\, f \rightarrowtail C$ is a split epi and hence an $\mathcal{H}$-morphism. Thus, $C \in \mathcal{H}\mathcal{H}\mathbf{V} = \mathcal{H}\mathbf{V}$ and so $A \in \mathcal{H}^{-}\mathcal{H}\mathbf{V}$. $\square$

We omit the proof of the following lemma, which is a consequence of Lemma 5 from [33].

Lemma 3.3 *For any composition* $\boldsymbol{X}$,

$$\mathbf{Proj}(K_{\boldsymbol{X}\Sigma}\mathbf{V}) = \mathbf{Proj}(K_{\boldsymbol{X}}\Sigma\mathbf{V}),$$
$$\mathbf{Proj}(K_{\boldsymbol{X}\Sigma^{+}}\mathbf{V}) = \mathbf{Proj}(K_{\boldsymbol{X}}\Sigma^{+}\mathbf{V}).$$

Theorem 3.4 *Let* $\mathbf{V}$ *be a full subcategory of* $\mathbb{C}$.

$$\mathbf{Proj}(K_{\mathcal{H}^{-}\mathcal{H}\Sigma}\mathbf{V}) = \mathcal{H}^{-}\mathcal{H}\Sigma\mathbf{V},$$
$$\mathbf{Proj}(K_{\mathcal{H}^{-}\mathcal{H}\Sigma^{+}}\mathbf{V}) = \mathcal{H}^{-}\mathcal{H}\Sigma^{+}\mathbf{V}.$$

Proof. By Lemmas 3.2 and 3.3. $\square$

Lemma 3.5 *Let* $\mathbf{V}$ *be a full subcategory of* $\mathbb{C}$.

$$\mathbf{Proj}(K_{\mathcal{H}^{-}\mathcal{S}\mathcal{H}}\mathbf{V}) = \mathcal{H}^{-}\mathcal{S}\mathcal{H}\mathbf{V}.$$

Proof. It suffices to show $\mathbf{Proj}(K_{\mathcal{H}^{-}\mathcal{S}\mathcal{H}}\mathbf{V}) \subseteq \mathcal{H}^{-}\mathcal{S}\mathcal{H}\mathbf{V}$. Let $A \models K_{\mathcal{H}^{-}\mathcal{S}\mathcal{H}}\mathbf{V}$ and consider

$$\mathcal{F}^{\mathbf{V}}1 = \{\mathsf{Im}(!_B) \rightarrowtail 1 \mid B \in \mathbf{V}\}.$$

Since $\mathbf{V} \models \langle 1, \mathcal{F}^{\mathbf{V}}1 \rangle$, also $A \models \langle 1, \mathcal{F}^{\mathbf{V}}1 \rangle$. Thus, $!_A : A \twoheadrightarrow 1$ factors through some $\mathsf{Im}(!_B)$, say, via g, as shown below.

$$\begin{tikzcd}
B \arrow[dr, two heads] & \mathsf{Im}\, g \arrow[d, tail] & A \arrow[l, two heads] \arrow[dl, "g"'] \arrow[d, "!_A"] \\
 & \mathsf{Im}(!_B) \arrow[r, tail] & 1
\end{tikzcd}$$

Because $\mathsf{Im}(!_B) \in \mathcal{H}\mathbf{V}$, we see $\mathsf{Im}(g) \in \mathcal{SH}\mathbf{V}$. Thus, $A \in \mathcal{H}^{-}\mathcal{SH}\mathbf{V}$. $\Box$

The following theorem is immediate from Lemmas 3.3 and 3.5.

Theorem 3.6 *Let* $\mathbf{V}$ *be a full subcategory of* $\mathbb{C}$.

$$\mathbf{Proj}(K_{\mathcal{H}^{-}\mathcal{SH}\Sigma}\mathbf{V}) = \mathcal{H}^{-}\mathcal{SH}\Sigma\mathbf{V},$$
$$\mathbf{Proj}(K_{\mathcal{H}^{-}\mathcal{SH}\Sigma^{+}}\mathbf{V}) = \mathcal{H}^{-}\mathcal{SH}\Sigma^{+}\mathbf{V}.$$

We have now proven, in an abstract setting, all of the generalized covariety theorems to be presented in this paper. Namely, we have shown that $\mathbf{Proj}(K_{\boldsymbol{X}}) = \boldsymbol{X}$ for every composition $\boldsymbol{X}$ considered herein (as described in Section 2.1). Note that this entails that each $\boldsymbol{X}$ is a closure operator. Before turning to interpreting these results in categories of coalgebras, we first want to motivate the definition of $\mathcal{H}^{-}$.

3.2 $\mathcal{H}^{-}$ and related operators

We are reluctant to add to the already long list of operators here, but it is worthwhile to consider two additional ones on $\mathfrak{P}(\mathsf{Ob}\,\mathbb{C})$ and describe $\mathcal{H}^{-}$ in terms of them. The goal is to motivate the description of $\mathcal{H}^{-}$ as a "behavioral" operator, at least once we focus on categories of coalgebras. Throughout the remainder of this section, we assume $\mathbb{C}$ has binary products, in order to facilitate the definition of $\mathcal{S}$-relation. Note that this entails SplitEpi $\subseteq \mathcal{H}$ "for free."

Definition 3.7 A triple $\langle R, r_1 : R \twoheadrightarrow A, r_2 : R \twoheadrightarrow B \rangle$ is an *$\mathcal{S}$-relation on A and B* just in case $\langle r_1, r_2 \rangle : R \rightarrowtail A \times B$ is in $\mathcal{S}$. An $\mathcal{S}$-relation $\langle R, r_1, r_2 \rangle$ on A and B is a *total $\mathcal{S}$-relation*, or just *total relation*, just in case r_1 and r_2 are in $\mathcal{H}$.

Let $\mathbb{C}$ be an arbitrary category with binary products, $\mathbf{V}$ a full subcategory of $\mathbb{C}$. Define two functors $\mathcal{R}$ and $\mathcal{T}$ as follows:

$$\mathcal{R}\mathbf{V} = \{B \in \mathbb{C} \mid \exists A \in \mathbf{V}\, \exists \text{ total relation } R \text{ on } A \text{ and } B\},$$
$$\mathcal{T}\mathbf{V} = \{B \in \mathbb{C} \mid \exists A \in \mathbf{V}\, \exists C\, \exists p : C \longrightarrow\!\!\!\!\!\rightarrow B,\ q : C \longrightarrow\!\!\!\!\!\rightarrow A\}.$$

In other words, $\mathcal{R}\mathbf{V}$ is the least class containing all B which are related by a total relation to some $A \in \mathbf{V}$, while $\mathcal{T}\mathbf{V}$ is the least class containing all B where there is a pair of $\mathcal{H}$-maps $p : C \twoheadrightarrow B$ and $q : C \twoheadrightarrow A$ with common domain and $A \in \mathbf{V}$ (but not necessarily $\langle p, q \rangle \in \mathcal{S}$). We may regard $\mathcal{T}\mathbf{V}$ as closure under *total multi-relations* [16]. The proof of the following lemma is straightforward.

Lemma 3.8 *$\mathcal{H}^-\mathcal{H}\mathbf{V} = \mathcal{R}\mathcal{R}\mathbf{V} = \mathcal{T}\mathcal{T}\mathbf{V}$ and if $\mathcal{H}$ is stable under pullbacks, then also $\mathcal{H}^-\mathcal{H}\mathbf{V} = \mathcal{R}\mathbf{V} = \mathcal{T}\mathbf{V}$.*

3.3 $\mathcal{H}^-$ dualized

We very briefly consider the dual of $\mathcal{H}^-$ and injectivity classes as in [33]. Since the focus of this section is the dual of Section 3.1, we will give few details here.

Let $\mathbb{C}$ have a factorization system $\langle \mathcal{H}, \mathcal{S}\rangle$, with $\mathcal{H} \subseteq$ Epi, products and an initial object 0. Further suppose that $\mathbb{C}$ is $\mathcal{H}$-well-co-powered and SplitMono $\subseteq \mathcal{S}$. Define operators

$$\begin{aligned}
\mathcal{S}^+(\mathbf{V}) &= \{B \in \mathbb{C} \mid \exists A \in \mathbf{V}\, \exists A \rightarrowtail B\},\\
\mathcal{P}(\mathbf{V}) &= \{\prod_{i\in I} A_i \mid A_i \in \mathbf{V}\},\\
\mathcal{P}^+(\mathbf{V}) &= \{\prod_{i\in I} A_i \mid A_i \in \mathbf{V} \text{ and } I \neq \emptyset\}.
\end{aligned}$$

The operators $\mathcal{P}$ and $\mathcal{P}^+$ appear in [33] and elsewhere, of course, but the operator $\mathcal{S}^+$ appears to be new in the context of Birkhoff-type projectivity theorems. In plain English, $\mathcal{S}^+$ is closure of $\mathbf{V}$ under codomains of abstract monos.

For a set T of discrete cones (note: cones, not cocones) in $\mathbb{C}$, we denote the injectivity class for T by $\mathbf{Inj}(T)$. Define the following operators taking subclasses of $\mathbb{C}$ to collections of cones.

$$\begin{aligned}
K_{\mathcal{H}}\mathbf{V} &= \{c = \langle C, \{f_i : C \longrightarrow B_i\}_i\rangle \mid \mathbf{V} \subseteq \mathbf{Inj}(\{c\}) \text{ and } C \text{ is } \mathcal{H}\text{-projective}\}\\
K_{\mathcal{S}}\mathbf{V} &= \{c = \langle C, \{f_i : C \twoheadrightarrow B_i\}_i\rangle \mid \mathbf{V} \subseteq \mathbf{Inj}(\{c\})\}\\
K_{\mathcal{S}^+}\mathbf{V} &= \{c = \langle C, \{f_i : C \longrightarrow B_i\}_i\rangle \mid \mathbf{V} \subseteq \mathbf{Inj}(\{c\}) \text{ and } \exists 0 \twoheadrightarrow C\}\\
K_{\mathcal{P}}\mathbf{V} &= \{c = \langle C, \{f_i : C \longrightarrow B_i\}_i\rangle \mid \mathbf{V} \subseteq \mathbf{Inj}(\{c\}) \text{ and } \mathbf{card}(I) = 1\}\\
K_{\mathcal{P}^+}\mathbf{V} &= \{c = \langle C, \{f_i : C \longrightarrow B_i\}_i\rangle \mid \mathbf{V} \subseteq \mathbf{Inj}(\{c\}) \text{ and } \mathbf{card}(I) \leq 1\}
\end{aligned}$$

For composites $\boldsymbol{X} = \mathcal{X}_1 \ldots \mathcal{X}_n$, define $K_{\boldsymbol{X}}\mathbf{V} = K_{\mathcal{X}_1}\mathbf{V} \cap \ldots \cap K_{\mathcal{X}_n}\mathbf{V}$.

Theorem 1 of ibid states (in part) that for each composite $\boldsymbol{X}$ in

$$\{\mathcal{S}, \mathcal{S}\mathcal{P}^+, \mathcal{S}\mathcal{P}, \mathcal{H}\mathcal{S}, \mathcal{H}\mathcal{S}\mathcal{P}^+, \mathcal{H}\mathcal{S}\mathcal{P}\},$$

we have $\boldsymbol{X}\mathbf{V} = \mathbf{Inj}(K_{\boldsymbol{X}}\mathbf{V})$. Dualizing Section 3.1, we get also that for any $\boldsymbol{X}$ as above,

$$\mathcal{S}^+\boldsymbol{X}\mathbf{V} = \mathbf{Inj}(K_{\mathcal{S}^+\boldsymbol{X}}\mathbf{V}).$$

If $\mathbb{C}$ is a category of algebras, this says, roughly, that $\mathbf{V}$ is closed under codomains of $\mathcal{S}$-morphisms iff $\mathbf{V}$ is definable by (implications between/Horn formulas of/etc.) equations with no variables, that is, equations involving ground terms.

4 Categories of coalgebras

Up to now, we have been working in an abstract category $\mathbb{C}$, assuming only i - vii from Section 3.1. Our real aim in this paper is to apply the previous results to categories of coalgebras. In this section, we show sufficient conditions that ensure a category of coalgebras satisfies our assumptions.

The following theorem was first proved in [3] for categories of algebras, so we omit the proof here. A similar theorem (in terms of inclusion systems) is found in [13].

Theorem 4.1 *Suppose that $\mathbb{C}$ has a factorization system $\langle \mathcal{H}, \mathcal{S} \rangle$. Suppose, further, that Γ preserves $\mathcal{S}$ (so for each $i \in \mathcal{S}$, $\Gamma i \in \mathcal{S}$). Then $\mathbb{C}_\Gamma$ has a factorization system $\langle U^{-1}\mathcal{H}, U^{-1}\mathcal{S} \rangle$.*

Since U creates coproducts, we see that if Γ preserves $\mathcal{S}$-morphisms, Theorem 2.3 applies to $\mathbb{C}_\Gamma$ for operators $\boldsymbol{X}$ not including the $\mathcal{S}$ operator. For operators $\mathcal{SH}$, $\mathcal{SH}\Sigma$, $\mathcal{SH}\Sigma^+$, we must ensure that $\mathbb{C}_\Gamma$ has enough injectives.

Theorem 4.2 *Suppose that $\mathbb{C}$ has enough $\mathcal{S}$-injectives and that $U : \mathbb{C}_\Gamma \twoheadrightarrow \mathbb{C}$ has a right adjoint $H : \mathbb{C} \twoheadrightarrow \mathbb{C}_\Gamma$. Then $\mathbb{C}_\Gamma$ has enough $\mathcal{S}$-injectives.*

Indeed, the proof for this proves that for every coalgebra $\langle A, \alpha \rangle$, there is a $U^{-1}\mathcal{S}$-morphism $\langle A, \alpha \rangle \rightarrowtail HC$, where C is $\mathcal{S}$-injective (and hence HC is $U^{-1}\mathcal{S}$-injective). Thus, if Γ preserves $\mathcal{S}$-morphisms and U has a right adjoint H, we can show

$$\mathcal{S}\boldsymbol{X}\mathbf{V} = \mathbf{Proj}(K_{\overline{\mathcal{S}}\boldsymbol{X}}\mathbf{V}),$$

where

$$K_{\overline{\mathcal{S}}}\mathbf{V} = \{c = \langle HC, \{f_i : \langle B_i, \beta_i \rangle \longrightarrow HC\}_{i \in I} \rangle \mid C \text{ is } \mathcal{S}\text{-injective and } \mathbf{V} \models c\}.$$

In other words, for $\mathcal{S}\boldsymbol{X}\mathbf{V}$, it suffices to consider cocones with a cofree vertex (specifically, cofree over some $\mathcal{S}$-injective).

Clearly, if SplitEpi $\subseteq \mathcal{H}$, then the split epis of $\mathbb{C}_\Gamma$ are contained in $U^{-1}\mathcal{H}$. Also, if $\mathbb{C}$ has a terminal object 1, and U has a right adjoint, then $H1$ is terminal in $\mathbb{C}_\Gamma$. Thus, if Γ preserves $\mathcal{S}$-morphisms and $\mathbb{C}_\Gamma$ has cofree coalgebras, then Theorems 3.4 and 3.6 apply. (More loosely, it suffices that $\mathbb{C}_\Gamma$ has a terminal object, however it arises, and that Γ preserves $\mathcal{S}$-morphisms.) Hence, we may apply the results of Section 3.1 to $\mathbb{C}_\Gamma$. Indeed, in many computer science applications of coalgebras, one is interested just in those classes which are closed under behavioral equivalence, and Section 3.1 contains the relevant results for these applications.

5 Cofree for $\boldsymbol{X}\mathbf{V}$ coalgebras

Throughout this section, we restrict our attention to compositions $\boldsymbol{X}$ involving both $\mathcal{H}$ and Σ^+ (or Σ). That is, we are interested in compositions

$$\boldsymbol{X} \in \{\mathcal{H}\Sigma^+,\, \mathcal{H}\Sigma,\, \mathcal{SH}\Sigma^+,\, \mathcal{SH}\Sigma,\, \mathcal{H}^-\mathcal{H}\Sigma^+,\, \mathcal{H}^-\mathcal{H}\Sigma,\, \mathcal{H}^-\mathcal{SH}\Sigma^+,\, \mathcal{H}^-\mathcal{SH}\Sigma\}.$$

This allows a simple construction of cocones $c^{\boldsymbol{X}\mathbf{V}}$ which suffice to define $\boldsymbol{X}\mathbf{V}$. Indeed, in this setting, we may focus on cocones with no more than one arrow.

The aim is to construct *cofree for* $\boldsymbol{X}\mathbf{V}$ *over* C coalgebras, at least for those C which permit such constructions. We begin by again returning to the abstract setting and considering full subcategories of arbitrary $\mathbb{C}$ satisfying i - v from Section 3.1. We define a subcategory $\mathbf{W} \hookrightarrow \mathbb{C}$ of $\mathbb{C}$ such that $\boldsymbol{X}\mathbf{V} \subseteq \mathbf{W}$ and furthermore $\boldsymbol{X}\mathbf{V}$ forms an $\mathcal{S}$-coreflection of $\mathbf{W}$. The construction of cofree for $\boldsymbol{X}\mathbf{V}$ coalgebras can then be viewed as a composition involving this coreflection.

Recall the definition $\mathcal{F}^{\mathbf{V}}A = \{\mathsf{Im}(f : B \twoheadrightarrow A) \mid B \in \mathbf{V}\}$ from Section 3.1. Define $\Delta^{\mathbf{V}} : \mathbb{C} \to \mathfrak{P}(\mathbb{C})$ by

$$\Delta^{\mathbf{V}}A = \bigvee \mathcal{F}^{\mathbf{V}}A.$$

Define $c^{\mathbf{V}} : \mathbb{C} \to \mathsf{Cocone}(\mathbb{C})$ by

$$c^{\mathbf{V}}B = \begin{cases} \langle B,\, \{\Delta^{\mathbf{V}}B \rightarrowtail B\}\rangle & \text{if } \mathcal{F}^{\mathbf{V}}B \neq \emptyset; \\ \langle B,\, \emptyset\rangle & \text{else.} \end{cases}$$

In case $\mathcal{F}^{\mathbf{V}}B \neq \emptyset$, we denote the $\mathcal{S}$-morphism $\Delta^{\mathbf{V}}B \rightarrowtail B$ by $\varepsilon^{\mathbf{V}}_B$.

Let $\mathbf{V} \subseteq \mathbb{C}$ and define

$$\mathrm{CoRe}(\mathbf{V}) = \{B \in \mathbb{C} \mid \mathcal{F}^{\mathbf{V}}B \neq \emptyset\}.$$

That is, $\mathrm{CoRe}(\mathbf{V})$ is the set of all $B \in \mathbb{C}$ such that there is some $A \in \mathbf{V}$ and arrow $A \twoheadrightarrow B$ in $\mathbb{C}$. Clearly, $\mathbf{V}$ is a subcategory of $\mathrm{CoRe}(\mathbf{V})$. Also, if $0 \in \mathbf{V}$ (say, if $\mathbf{V} = \Sigma\mathbf{V}$), then $\mathrm{CoRe}(\mathbf{V}) = \mathbb{C}$. In this case, as the next theorem shows, $\mathbf{V}$ is an $\mathcal{S}$-coreflective subcategory of $\mathbb{C}$. Indeed, if $\mathbf{V}$ is closed under isomorphisms, then $\mathbf{V} = \mathcal{H}\Sigma\mathbf{V}$ iff $\mathbf{V}$ is a $\mathcal{S}$-coreflective subcategory of $\mathbb{C}$.

Theorem 5.1 *The inclusion* $\boldsymbol{X}\mathbf{V} \hookrightarrow \mathrm{CoRe}(\boldsymbol{X}\mathbf{V})$ *has right adjoint* $\Delta^{\boldsymbol{X}\mathbf{V}}$*, with counit* $\varepsilon^{\boldsymbol{X}\mathbf{V}}$*. In particular,* $\boldsymbol{X}\mathbf{V} \hookrightarrow \mathrm{CoRe}(\boldsymbol{X}\mathbf{V})$ *is an* $\mathcal{S}$*-coreflection, in the sense that each component of the counit* $\varepsilon^{\boldsymbol{X}\mathbf{V}}$ *is in* $\mathcal{S}$*.*

Proof. For any $B \in \mathrm{CoRe}(\boldsymbol{X}\mathbf{V})$, we have $\Delta^{\boldsymbol{X}\mathbf{V}}B \in \mathcal{H}\Sigma^+\boldsymbol{X}\mathbf{V} = \boldsymbol{X}\mathbf{V}$, so $\Delta^{\boldsymbol{X}\mathbf{V}}$ is indeed a functor $\mathrm{CoRe}(\boldsymbol{X}\mathbf{V}) \twoheadrightarrow \boldsymbol{X}\mathbf{V}$ as desired. It is straightforward to confirm that $\varepsilon^{\boldsymbol{X}\mathbf{V}}$ satisfies the universal mapping property for counits. $\square$

Finally, we return to the setting $\mathbb{C}_\Gamma$, where Γ preserves $\mathcal{S}$-morphisms and $U : \mathbb{C}_\Gamma \to \mathbb{C}$ has a right adjoint $H : \mathbb{C} \to \mathbb{C}_\Gamma$. First, we define cofree for $\mathbf{V}$ over C coalgebras and then apply the previous construction to $\mathbb{C}_\Gamma$.

Definition 5.2 Let $\langle A, \alpha\rangle \in \mathbb{C}_\Gamma$ and $p : A \to C$ be given and let $\mathbf{V}$ be a subcategory of $\mathbb{C}_\Gamma$. We say that $\langle A, \alpha\rangle$ (with the "coloring" p) is *cofree for* $\mathbf{V}$ *over* C just in case

- $\langle A, \alpha\rangle \in \mathbf{V}$;
- for every $\langle B, \beta\rangle \in \mathbf{V}$ and $q : B \to C$, there is a unique Γ-homomorphism $\widetilde{q} : \langle B, \beta\rangle \to \langle A, \alpha\rangle$ such that the diagram below commutes.

$$\begin{tikzcd} & A \arrow[d, "p"] \\ B \arrow[ur, "\widetilde{q}"] \arrow[r, "q"'] & C \end{tikzcd}$$

Let $\mathbf{V} \subseteq \mathbb{C}_\Gamma$ and define

$$\exists_U \mathbf{V} = \{C \in \mathbb{C} \mid \exists \gamma : C \longrightarrow \Gamma C \,.\, \langle C, \gamma\rangle \in \mathbf{V}\}.$$

That is, $\exists_U \mathbf{V}$ is the image of $\mathbf{V}$ under the forgetful functor U.

Theorem 5.3 *For any $C \in \mathrm{CoRe}(\exists_U \boldsymbol{X}\mathbf{V})$, there is a cofree for $\boldsymbol{X}\mathbf{V}$ over C coalgebra. In particular, if $\mathbb{C} = \mathbf{Set}$, then there is a cofree for $\boldsymbol{X}\mathbf{V}$ over C coalgebra for any $C \neq \emptyset$. Also, if $\langle 0, !\rangle \in \boldsymbol{X}\mathbf{V}$, then for every C, there is a cofree for $\boldsymbol{X}\mathbf{V}$ over C coalgebra.*

Proof. We note that $C \in \mathrm{CoRe}(\exists_U \boldsymbol{X}\mathbf{V})$ iff there is a coalgebra $\langle B, \beta\rangle$ in $\mathbf{V}$ and a map $q : B \to C$. Hence, $C \in \mathrm{CoRe}(\exists_U \boldsymbol{X}\mathbf{V})$ iff $HC \in \mathrm{CoRe}(\boldsymbol{X}\mathbf{V})$. Let

$$\begin{tikzcd} \boldsymbol{X}\mathbf{V} \arrow[r, bend left, "I"] \arrow[r, phantom, "\perp"] & \mathrm{CoRe}(\boldsymbol{X}\mathbf{V}) \arrow[l, bend left, "\Delta"] \end{tikzcd}$$

be the $\mathcal{S}$-coreflection with counit $\varepsilon^{\boldsymbol{X}\mathbf{V}}$, as in Theorem 5.1. Then the cofree for $\boldsymbol{X}\mathbf{V}$ over C coalgebra (for $C \in \mathrm{CoRe}(\exists_U \boldsymbol{X}\mathbf{V})$) is given by $UI\Delta HC$, with coloring $\varepsilon_C \circ U\varepsilon^{\boldsymbol{X}\mathbf{V}}_{HC} : UI\Delta HC \to C$. $\square$

6 Some examples of Horn covarieties

We close with some examples of so-called Horn covarieties, i.e., classes $\mathbf{V}$ of coalgebras such that $\mathbf{V} = \mathcal{H}\Sigma^+\mathbf{V}$.

There is a sense in which the distinction between quasi-covarieties ($\mathbf{V} = \mathcal{H}\Sigma\mathbf{V}$) and Horn covarieties is trivial. Namely, a (proper) Horn covariety is just a quasi-covariety which omits the trivial coalgebra $\langle 0, !\rangle$. In our examples, we hope to show that some natural classes of coalgebras arise as Horn covarieties, the apparent triviality of the distinction between $\mathcal{H}\Sigma$ and $\mathcal{H}\Sigma^+$ notwithstanding.

Example 6.1 Fix a non-empty set Z and let $\Gamma : \mathbf{Set} \to \mathbf{Set}$ be the functor $\Gamma X = Z \times X$. For a given coalgebra $\langle A, \alpha\rangle$, we let $h_\alpha = \pi_1 \circ \alpha$ and $t_\alpha = \pi_2 \circ \alpha$.

Let $\mathbf{V} \subseteq \mathbf{Set}_\Gamma$ be the class

$$\{\langle A,\, \alpha\rangle \in \mathbf{Set}_\Gamma \mid \exists a \in A \,.\, t_\alpha(a) = a\}.$$

Then $\mathbf{V}$ is a proper Horn covariety (i.e., $\mathbf{V} = \mathcal{H}\Sigma^+\mathbf{V} \neq \mathcal{H}\Sigma\mathbf{V}$). The class

$$\mathbf{W} = \{\langle A,\, \alpha\rangle \in \mathbf{Set}_\Gamma \mid \exists a \in A \forall n \in \mathbb{N} \,.\, h_\alpha(a) = h_\alpha \circ t_\alpha^n(a)\}$$

is a behavioral Horn covariety ($\mathbf{W} = \mathcal{H}^-\mathcal{H}\Sigma^+\mathbf{W}$).

Remark 6.2 Many of the examples of Horn covarieties we know (including Example 6.1, above) are closed under codomains of arbitrary maps, rather than just coproducts and codomains of epis. One wonders whether collections closed under codomains of *arbitrary* morphisms have a characterization along the lines of Theorem 2.2. That is, if we define $\mathcal{M}\mathbf{V} = \{A \in \mathbb{C} \mid \exists A \rightarrow B \text{ and } B \in \mathbf{V}\}$, is there a functor $K_\mathcal{M} : \mathfrak{P}(\mathbb{C}) \rightarrow \mathfrak{P}(\mathsf{Cocone}(\mathbb{C}))$ such that, for each $\mathbf{V}$, $K_\mathcal{M}\mathbf{V} \subsetneqq K_{\mathcal{H}\Sigma^+}\mathbf{V}$ and $\mathcal{M}\mathbf{V} = \mathbf{Proj}(K_\mathcal{M}\mathbf{V})$? We leave this question for future study.

Example 6.3 Fix an alphabet $\mathcal{I}$. Let $\Gamma : \mathbf{Set} \rightarrow \mathbf{Set}$ be the functor $X \mapsto 2 \times X^\mathcal{I}$. We interpret Γ-coalgebras as automata which accept as input finite words over $\mathcal{I}$ and output 0 or 1. Explicitly, given $\langle A,\, \alpha\rangle \in \mathbf{Set}_\Gamma$, $a \in A$, let

$$\begin{aligned} \mathsf{out}_\alpha(a) &= \pi_1 \circ \alpha(a), \\ \mathsf{trans}_\alpha(a) &= \pi_2 \circ \alpha(a). \end{aligned}$$

It is routine to show that, given $\langle A,\, \alpha\rangle$ and $\langle B,\, \beta\rangle$, a map $f : A \rightarrow B$ is a homomorphism just in case, for every $a \in A$, $i \in \mathcal{I}$,

$$\begin{aligned} \mathsf{out}_\beta(f(a)) &= \mathsf{out}_\alpha(a), \\ \mathsf{trans}_\beta(f(a))(i) &= f(\mathsf{trans}_\alpha(a)(i)). \end{aligned}$$

We let σ range over $\mathcal{I}^{<\omega}$ and let $\sigma * i$ denote the word formed by adjoining i at the end of σ. Define $\mathsf{eval}_\alpha : A \times \mathcal{I}^{<\omega} \rightarrow A$ by

$$\begin{aligned} \mathsf{eval}_\alpha(a, ()) &= a, \\ \mathsf{eval}_\alpha(a, \sigma * i) &= \mathsf{trans}_\alpha(\mathsf{eval}_\alpha(a, \sigma))(i). \end{aligned}$$

The state $\mathsf{eval}_\alpha(a, \sigma)$ is the final state reached when one executes the transitions given by σ, beginning in state a.

Define $\mathsf{acc}_\alpha : A \rightarrow \mathcal{P}(\mathcal{I}^{<\omega})$ by

$$\mathsf{acc}_\alpha(a) = \{\sigma \in \mathcal{I}^{<\omega} \mid \mathsf{out}_\alpha \circ \mathsf{eval}_\alpha(a, \sigma) = 1\}.$$

Intuitively, $\mathsf{acc}_\alpha(a)$ are all of the words over $\mathcal{I}$ which $\langle A,\, \alpha\rangle$ accepts from initial state a.

Fix a "language" $\mathcal{L} \subseteq \mathcal{I}^{<\omega}$ and define

$$\mathbf{V}_{\mathcal{L}} = \{\langle A,\, \alpha\rangle \in \mathbf{Set}_{\Gamma} \mid \exists a \in A\,.\; \mathsf{acc}_{\alpha}(a) = \mathcal{L}\}.$$

We claim that $\mathbf{V}_{\mathcal{L}}$ is a behavioral Horn covariety. Stronger, we claim that $\mathbf{V}_{\mathcal{L}}$ is closed under codomains of arbitrary homomorphisms (as well as domains of epis). We omit the proof of the following lemma.

Lemma 6.4 *For every $f : \langle A,\, \alpha\rangle \twoheadrightarrow \langle B,\, \beta\rangle$ and $a \in A$,*

$$\mathsf{eval}_{\beta}(f(a), -) = f(\mathsf{eval}_{\alpha}(a, -)).$$

Suppose $\langle A,\, \alpha\rangle \in \mathbf{V}_{\mathcal{L}}$ and let $f : \langle A,\, \alpha\rangle \twoheadrightarrow \langle B,\, \beta\rangle$ be given. Let $a \in A$ such that $\mathsf{acc}_{\alpha}(a) = \mathcal{L}$. Then, since

$$\mathsf{out}_{\beta} \circ \mathsf{eval}_{\beta}(f(a), \sigma) = \mathsf{out}_{\beta} \circ f \circ \mathsf{eval}_{\alpha}(a, \sigma) = \mathsf{out}_{\alpha} \circ \mathsf{eval}_{\alpha}(a, \sigma),$$

we see that $\mathsf{acc}_{\alpha}(a) = \mathsf{acc}_{\beta}(f(a))$ and so $\langle B,\, \beta\rangle \in \mathbf{V}_{\mathcal{L}}$. Similarly, if f is epi (and hence onto), we see that $\langle B,\, \beta\rangle \in \mathbf{V}_{\mathcal{L}}$ implies $\langle A,\, \alpha\rangle \in \mathbf{V}_{\mathcal{L}}$. That is, $\mathbf{V}_{\mathcal{L}} = \mathcal{H}^{-}\mathcal{H}\Sigma\mathbf{V}_{\mathcal{L}}$.

We may describe a defining cocone for $\mathbf{V}_{\mathcal{L}}$ quite easily. It can be given as an $\mathcal{S}$-morphism into $H1$, the final coalgebra (see Section 3.1). The final coalgebra $H1$ can be described as $\mathcal{P}(\mathcal{I}^{<\omega})$, the set of all languages over $\mathcal{I}$, together with the structure map

$$\mathcal{P}(\mathcal{I}^{<\omega}) \longrightarrow 2 \times \mathcal{P}(\mathcal{I}^{<\omega})^{\mathcal{I}}$$

given by $\mathsf{out}(\mathcal{L}') = 1$ iff $() \in \mathcal{L}'$ and $\mathsf{trans}(\mathcal{L}')(i) = \{\sigma \mid i * \sigma \in \mathcal{L}'\}$ (where $i * \sigma$ is the word formed by prepending i to σ). Note that for every $\mathcal{L}'$, we have $\mathsf{acc}(\mathcal{L}') = \mathcal{L}'$. It is easy to see that a coalgebra $\langle A,\, \alpha\rangle$ is in $\mathbf{V}_{\mathcal{L}}$ iff the homomorphism $!_A : \langle A,\, \alpha\rangle \twoheadrightarrow H1$ does *not* factor through the largest subcoalgebra contained in the subobject

$$\varphi_{\mathcal{L}} = \{\mathcal{L}' \in \mathcal{P}(I^{<\omega}) \mid \mathcal{L}' \neq \mathcal{L}\}.$$

That is, $\langle A,\, \alpha\rangle \in \mathbf{V}_{\mathcal{L}}$ just in case $\langle A,\, \alpha\rangle$ is projective with respect to the empty cocone $\langle \Box\varphi_{\mathcal{L}},\, \emptyset\rangle$, where $\Box\varphi_{\mathcal{L}}$ is the largest subcoalgebra of $H1$ contained in $\varphi_{\mathcal{L}}$.

We close this section with a table showing some interesting classes of automata and the closure conditions they satisfy. This table is suggests that there is a relationship between closure conditions on a class of coalgebras and the first-order formula defining the class, but this relationship is far from clear.

Incidentally, each of the classes shown below is definable by a single cocone of the appropriate shape. That is, for each $\mathbf{V}$ below, there is a cocone c such that $\mathbf{V} = \mathbf{Proj}(\{c\})$. We omit explicitly giving these cocones here, as it requires some development to describe them.

$\mathbf{V} = \{\langle A, \alpha\rangle \mid \ldots\}$	$\mathbf{V}$ closed under
$\forall a \in A\,.\ \mathsf{acc}(a) \neq \mathcal{L}$	$\mathcal{H}^{-}\mathcal{S}\mathcal{H}\Sigma$
$A \neq \emptyset \implies \exists a \in A\,.\ \mathsf{acc}(a) = \mathcal{L}$	$\mathcal{H}^{-}\mathcal{H}\Sigma$
$\exists a \in A\,.\ \mathsf{acc}(a) = \mathcal{L}$	$\mathcal{H}^{-}\mathcal{H}\Sigma^{+}$
$\exists_{\leq 1} a \in A\,.\ \mathsf{acc}(a) = \mathcal{L}$	$\mathcal{S}\mathcal{H}$
$\exists! a \in A\,.\ \mathsf{acc}(a) = \mathcal{L}$	$\mathcal{H}$
$\exists! a \in A\,.\ \mathsf{acc}(a) = \mathcal{L}$ and $\forall b \in A\,.\ b \xrightarrow{*} a$	$\mathcal{S}\mathcal{H}$

The formula $\exists! a\psi$ expresses that there is a unique a satisfying ψ, and $\exists_{\leq 1} a\psi$ expresses that there is at most one such a.

7 Conclusion and future topics

The main aim of this paper is to interpret the theorems of Németi and Sain for categories of coalgebras and also to extend these theorems by considering the $\mathcal{H}^{-}$ operator, which arises naturally in applications of co-Birkhoff-type theorems in computer science. This extension allowed some new Birkhoff-type results by duality. We also gave a construction of cofree for $\boldsymbol{X}\mathbf{V}$ coalgebras in Section 6 for several compositions $\boldsymbol{X}$. Lastly, we closed with some examples of classes $\mathbf{V}$ of coalgebras where $\mathbf{V} = \mathcal{H}\Sigma^{+}\mathbf{V}$, in order to justify the claim that "Horn covarieties" arise naturally in categories of coalgebras, and in particular in those categories of interest in computer science.

The co-Birkhoff-type theorems here are really only half of the picture. One might ask what deductive completeness theorems arise from the closure operators studied here. A literal dualization of Birkhoff's deductive completeness theorem leads to a pair of modal operators investigated in [22]. Recent work shows how these operators lead to rules of inference for sets of coequations that are complete with respect to coequation satisfaction. We also have a deductive completeness theorem for conditional coequations. However, both theories are logics only in an abstract sense, and the result is not satisfying. Further work is needed here to understand the situation and see whether a more reasonable logic is attainable in a restricted setting.

There is another open question involving logic and projectivity classes. The classes $\mathbf{V}$ of coalgebras in which we are typically interested are defined by

$$\mathbf{V} = \{\langle A, \alpha\rangle \in \mathbb{C}_{\Gamma} \mid P(\langle A, \alpha\rangle)\},$$

where P is a proposition involving terms related to the functor Γ. One would like to know if there is a connection between the structure of P and the closure conditions $\mathbf{V}$ satisfies. Can we determine from the syntax of P that

$\mathbf{V} = \mathcal{H}\Sigma^{+}\mathbf{V}$ (say)? The answer to this question may be related to [8], but it is not obvious that the relationship between first order logic and injectivity classes leads to a similar relationship between first order logic and projectivity classes.

One would also like to investigate the operator taking a collection $\mathbf{V}$ to its closure of domains under arbitrary homomorphisms, as mentioned in Remark 6.2. And lastly, but importantly, what is the appropriate notion of "finitary" classes of coalgebras that takes the place of closure under ultraproducts for classes of algebras? And what are the appropriate conditions on $\mathbb{C}$ and Γ that ensure finitary versions of Theorem 2.3 go through?

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