# THE COALGEBRAIC DUAL OF BIRKHOFF'S VARIETY THEOREM

STEVE AWODEY AND JESSE HUGHES

Abstract. We prove an abstract dual of Birkhoff's variety theorem for categories $\mathcal{E}_\Gamma$ of coalgebras, given suitable assumptions on the underlying category $\mathcal{E}$ and suitable $\Gamma : \mathcal{E} \longrightarrow \mathcal{E}$. We also discuss covarieties closed under bisimulations and show that they are definable by a trivial kind of coequation – namely, over one "color". We end with an example of a covariety which is *not* closed under bisimulations.

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## 1. Introduction

One of the earliest theorems in universal algebra is Garrett Birkhoff's Variety Theorem [Bir35]. It states that a class $\mathbf{V}$ of algebras is closed under homomorphic images, subalgebras and products just in case $\mathbf{V}$ is the collection of all algebras satisfying some set of equations.

The classical definition of $\Sigma$-algebras for a signature $\Sigma$ generalizes to the category theoretic notion of $\Gamma$-algebras for an endofunctor $\Gamma$. This, in turn, leads to the dual notion of $\Gamma$-coalgebras. Coalgebras have proven useful in modeling processes and objects in computer science [JR97]. It is natural to ask whether such a basic theorem as Birkhoff's variety theorem can be dualized to a "Co-Birkhoff Covariety Theorem" for coalgebras.

In order to dualize the Birkhoff Variety Theorem, we first prove a general variety theorem in an abstract setting, showing that it holds for a wide range of categories. This theorem also yields the Birkhoff theorem in the classical setting.

The principle of categorical duality then yields two covariety theorems. On the one hand, the abstract version of the classical theorem leads to an abstract covariety theorem that applies to all categories that satisfy some basic factorization properties. On the other hand, this abstract version specializes to the following theorem for certain categories of coalgebras:

**Theorem.** *A full subcategory* $\mathbf{V}$ *of a category of coalgebras is closed under images, subcoalgebras and coproducts iff* $\mathbf{V}$ *is the class of all coalgebras that satisfy a collection of coequations.*

The notion of a "coequation" is determined by duality, and can be regarded as a condition on the possible "colorings" of a coalgebra.

The covarieties over **Set** which are closed under bisimulations (here called *behavioral covarieties*) are studied in [GS98]. We generalize this work and provide a natural example of a covariety which is not behavioral.

A dual to Birkhoff's variety theorem for coalgebras over **Set** was first mentioned in [Rut96]. This result was further developed in [GS98], where behavioral covarieties were first studied (under the name *complete covarieties*). Behavioral covarieties also arise (under the name *sinks*) in [Roş00]. We take a more general approach, which yields a covariety theorem for coalgebras over a wide class of categories. The basic approach was first developed for varieties in [BH76], which we discovered after proving the results herein. Alexander Kurz simultaneously and independently took the same approach for covarieties and proved similar results in [Kur00].

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## 2. The Classical Theorem from a Categorical Perspective

Birkhoff's theorem considers $\Sigma$-algebras satisfying a set of equations. We take the perspective here that a $\Sigma$-algebra is an algebra for a polynomial endofunctor, and equational definability is a special kind of orthogonality condition. Alternatively, in the terminology of [AR94], a class of algebras is equationally definable just in case it is an injectivity class for an appropriate collection of morphisms.

In more details, given a signature $\Sigma$, the category of $\Sigma$-algebras is isomorphic to the category $\mathbf{Set}^{\mathbb{P}}$ of $\mathbb{P}$-algebras

$$\langle A,\, \alpha : \mathbb{P}A \longrightarrow A \rangle$$

for some polynomial functor $\mathbb{P}:\mathbf{Set}\longrightarrow\mathbf{Set}$,

$$\mathbb{P}A = C_0 + C_1 \times A + \ldots + C_n \times A^n.$$

As is well known, the forgetful functor

$$U:\mathbf{Set}^{\mathbb{P}}\longrightarrow\mathbf{Set}$$

has a left adjoint

$$F:\mathbf{Set}\longrightarrow\mathbf{Set}^{\mathbb{P}}$$

taking a set $A$ to the free $\mathbb{P}$-algebra over $A$. This algebra is the term algebra over $A$ and is described categorically as the initial $A+\mathbb{P}(-)$-algebra (viewed as a $\mathbb{P}$ algebra by forgetting part of the structure map).

An equation $\tau_1 = \tau_2$ over variables in the set $X$ is just a pair of elements of $FX$,

$$\tau_1, \tau_2 : F1\longrightarrow FX.$$

An algebra $\langle A, \alpha\rangle$ satisfies $\tau_1 = \tau_2$ iff, for every assignment of the variables

$$\sigma : X\longrightarrow A,$$

the unique homomorphic extension $\overline{\sigma}:FX\longrightarrow\langle A, \alpha\rangle$ coequalizes $\tau_1$ and $\tau_2$. Let

$$F1 \underset{\widetilde{\tau_2}}{\overset{\widetilde{\tau_1}}{\rightrightarrows}} FX \xrightarrow{q} Q \tag{1}$$

be a coequalizer diagram[1]. Then, we have that $\langle A, \alpha\rangle \models \tau_1 = \tau_2$ iff every homomorphism $FX\longrightarrow\langle A, \alpha\rangle$ factors (necessarily uniquely) through $q$.

2.1. **Orthogonality.** In general, a map $f:A\longrightarrow B$ is said to be *orthogonal* to an object $C$ (written $f \perp C$) if, for every map $a:A\longrightarrow C$, there is a unique map $b:B\longrightarrow C$ such that $a = b\circ f$, i.e., $\mathsf{Hom}(f, C)$ is an isomorphism (see, for example, [Bor94, Volume 1, Section 5.4]). Thus, we see that an algebra $\langle A, \alpha\rangle$ satisfies $\tau_1 = \tau_2$ just in case $q \perp \langle A, \alpha\rangle$, where $q$ is the coequalizer of $\widetilde{\tau_1}$ and $\widetilde{\tau_2}$, as in (1). Furthermore, for any regular epi $p$ with domain $FX$, there is clearly a set of equations $E$ over $X$ such that $\langle A, \alpha\rangle \models E$ iff $p \perp \langle A, \alpha\rangle$. Thus, orthogonality is a generalization of satisfaction of equations. We intend to understand Birkhoff's variety theorem as a theorem that relates certain closure conditions to orthogonality.

If $S$ is a collection of arrows of $\mathcal{C}$, we write $S \perp C$ if $f \perp C$ for all $f \in S$. Similarly, if $\mathbf{V}$ is a collection of objects of $\mathcal{C}$ (regarded as a full subcategory), we

[1] We use $\bullet\!\!\longrightarrow\!\!\!\triangleright\bullet$ to denote a regular epimorphism and $\bullet\!\triangleright\!\!\!\longrightarrow\bullet$ to denote a regular monomorphism.

write $f \perp \mathbf{V}$ if $f \perp C$ for each $C \in \mathbf{V}$. Finally, we define the notation $S \perp \mathbf{V}$ in the obvious way. $S^{\perp}$ is the collection of all objects $C$ such that $S \perp C$. Similarly, $\mathbf{V}^{\perp}$ is the collection of all arrows $f$ such that $f \perp \mathbf{V}$.

The class $\mathsf{Sub}(\mathcal{C}_1)$ of all collections of maps of $\mathcal{C}$ is partially ordered under inclusion. Similarly, the class $\mathsf{Sub}(\mathcal{C}_0)$ of all full subcategories of $\mathcal{C}$ is also a partial order under inclusion. Since $S \subseteq T$ implies $S^{\perp} \supseteq T^{\perp}$ for sets of arrows, and for subcategories $\mathbf{V} \subseteq \mathbf{W}$ implies $\mathbf{W}^{\perp} \subseteq \mathbf{V}^{\perp}$, we have contravariant functors:

$$\mathsf{Sub}(\mathcal{C}_1) \rightleftarrows (\mathsf{Sub}\,\mathcal{C}_0)^{\mathsf{op}}.$$

Moreover, it is easy to see that, given a collection of maps $S$ and a full subcategory $\mathbf{V}$, $S^{\perp} \subseteq \mathbf{V}$ iff $S \supseteq \mathbf{V}^{\perp}$. Thus, the two $\perp$ functors form a Galois correspondence, and $\perp\perp$ is a closure operation (see [Bor94, Volume 1, Example 3.1.6.m]).

In terms of orthogonality, then, we can state Birkhoff's variety theorem thus:

**Theorem 2.1.** *Let $\mathbb{P}$ be a polynomial functor and $\mathbf{V}$ a full subcategory of $\mathbf{Set}^{\mathbb{P}}$. Let $X$ be an infinite set. Then $\mathbf{V}$ is closed under quotients, subalgebras and products iff $\mathbf{V} = \{q\}^{\perp}$ for some regular epi $q$ with domain $FX$.*

2.2. **An abstract variety theorem.** We begin by proving an abstract variety theorem which entails the classical Birkhoff variety theorem.

Recall that a category $\mathcal{C}$ is *regularly co-well-powered* if, for each object $C$ in $\mathcal{C}$, the collection of (isomorphism classes of) regular epis with domain $C$ is a set. A category *has enough projectives* if, for each $C$, there is a projective[2] object $A$ and a regular epi $A \twoheadrightarrow C$. In the abstract theorem, the projective objects will play the role of $FX$, the free algebra over a set of variables.

**Definition 2.1.** A *quasi-Birkhoff category* is a category that is complete, regularly co-well-powered and has regular epi-mono factorizations. A *Birkhoff category* is a quasi-Birkhoff category with enough projectives.

*Example* 2.1. Let $\mathcal{C}$ be a category with finite limits and $C$ in $\mathcal{C}$. Then regular epis $q : C \twoheadrightarrow \bullet$ give rise to subobjects of $C \times C$ (take the kernel pair of $q$ and regard it as a subobject of $C \times C$). Hence, if $\mathcal{C}$ is complete, has regular-epi mono factorizations and is well-powered, then $\mathcal{C}$ is a quasi-Birkhoff category.

[2]Throughout, we are interested in objects that are projective for regular epis, rather than for every epi. These are sometimes called "regular projective" objects.

*Example* 2.2. The category of algebras for a monad on **Set** is always Birkhoff.

Let $\mathcal{C}$ be a complete category. Given a full subcategory $\mathbf{V}$ of $\mathcal{C}$, we say that $\mathbf{V}$ is *closed under subobjects* if, whenever there is a monic $A \rightarrowtail B$ and $B \in \mathbf{V}$, then $A \in \mathbf{V}$. We say that $\mathbf{V}$ is *closed under quotients* if, whenever $A \longrightarrow\!\!\triangleright B$ is a regular epi and $A \in \mathbf{V}$, then $B \in \mathbf{V}$. We say that $\mathbf{V}$ is *closed under products* if the inclusion functor, $U^{\mathbf{V}} : \mathbf{V} \longrightarrow \mathcal{C}$, creates products.

*Remark* 2.3. Let $\mathbf{V}$ be a full subcategory of $\mathcal{C}$, closed under subobjects. Then $U^{\mathbf{V}}$ creates limits iff $\mathbf{V}$ is closed under products.

**Definition 2.2.** Let $\mathcal{C}$ be quasi-Birkhoff. A *Birkhoff quasi-variety of* $\mathcal{C}$ is a full subcategory of $\mathcal{C}$ closed under subobjects and products. If $\mathcal{C}$ is a Birkhoff category, a *Birkhoff variety* is a Birkhoff quasi-variety closed under quotients.

**Theorem 2.2.** *Let $\mathcal{C}$ be a quasi-Birkhoff category and $\mathbf{V}$ a full subcategory of $\mathcal{C}$. The following are equivalent.*

1. $\mathbf{V}$ *is closed under products and subobjects (i.e., $\mathbf{V}$ is a quasi-variety).*
2. $\mathbf{V}$ *is a regular epi-reflective subcategory of $\mathcal{C}$. That is, a subcategory whose inclusion $U^{\mathbf{V}} : \mathbf{V} \longrightarrow \mathcal{C}$ has a left adjoint $F^{\mathbf{V}}$ such that each component of the unit $\eta^{\mathbf{V}} : U^{\mathbf{V}} F^{\mathbf{V}} \longrightarrow 1_{\mathcal{C}}$ is a regular epi.*
3. $\mathbf{V} = S^{\perp}$ *for some collection $S$ of regular epis.*

*Proof.* **$1 \Rightarrow 2$:** By the adjoint functor theorem (see, for example, [Bor94, Volume 1,Theorem 3.3.3]). The solution set condition is satisfied by the (set-many) quotients of each object. Because $\mathbf{V}$ is closed under subobjects, the reflection is a regular epireflection ([Bor94, Volume 1, Proposition 3.6.4]).

**$2 \Rightarrow 3$:** One shows that $(\eta^{\mathbf{V}})^{\perp} = \mathbf{V}$.

**$3 \Rightarrow 1$:** Easy.

□

**Corollary 2.3.** *Let $\mathcal{C}$ be a quasi-Birkhoff category and $\mathbf{V}$ a quasi-variety of $\mathcal{C}$. Then*

1. *The inclusion $U^{\mathbf{V}} : \mathbf{V} \longrightarrow \mathcal{C}$ has a left adjoint $F^{\mathbf{V}}$.*
2. *The counit $\varepsilon^{\mathbf{V}} : F^{\mathbf{V}} U^{\mathbf{V}} \longrightarrow \mathsf{id}_{\mathbf{V}}$ is an isomorphism.*

3. *For each $C \in \mathcal{C}$, $C \in \mathbf{V}$ iff $\eta^{\mathbf{V}}_C \perp C$, where $\eta^{\mathbf{V}}$ is the unit of the adjunction $F^{\mathbf{V}} \dashv U^{\mathbf{V}}$.*
4. *The corresponding monad, $\mathbb{T}^{\mathbf{V}} = U^{\mathbf{V}} F^{\mathbf{V}}$, is idempotent.*
5. *The monad $\mathbb{T}^{\mathbf{V}}$ preserves regular epis.*

*Example* 2.4. **Set** is quasi-Birkhoff. However, the only quasi-varieties of **Set** are trivial. Indeed, let $\mathbf{V}$ be a quasi-variety. If $2 \in \mathbf{V}$, then $2^\alpha$ is in $\mathbf{V}$ for every ordinal $\alpha$. Since $\mathbf{V}$ is closed under subobjects, we have that $\mathbf{V} = \mathbf{Set}$. If $2 \notin \mathbf{V}$, then $\mathbf{V}$ must consist of just 0 and 1.

*Example* 2.5. The category **Mon** of monoids is complete, regular and well-powered, hence, a quasi-Birkhoff category. Let $\mathbf{V}$ be the subcategory of **Mon** consisting of those monoids satisfying

$$x^2 = e \to x = e.$$

Then $\mathbf{V}$ is clearly closed under subalgebras and products. Thus, by Theorem 2.2, $\mathbf{V}$ is a regular epi-reflective subcategory of **Mon**.

**Theorem 2.4.** *If $\mathcal{C}$ is a Birkhoff category, then a full subcategory $\mathbf{V}$ is a variety iff $\mathbf{V} = S^\perp$ for some collection $S$ of regular epis with projective domains.*

*Proof.* Let $\mathbf{V}$ be a variety. For each $C$ in $\mathcal{C}$, let $A_C$ be the projective that covers $C$. Let $S$ be the collection of all

$$\eta^{\mathbf{V}}_{A_C} : A_C \longrightarrow U^{\mathbf{V}} F^{\mathbf{V}} A_C \,,$$

where $\eta^{\mathbf{V}}$ is the unit of the adjunction from Theorem 2.2. We will show that $\mathbf{V} = S^\perp$.

If $\eta^{\mathbf{V}}_{A_C} \perp C$, then there is a unique map $\overline{p_C}$ making the diagram below commute.

$$\begin{tikzcd}
A_C \arrow[r, "p_C"] \arrow[d, "\eta^{\mathbf{V}}_{A_C}"'] & C \\
U^{\mathbf{V}} F^{\mathbf{V}} C \arrow[ur, "\overline{p_C}"'] &
\end{tikzcd}$$

Since $\overline{p_C}$ is a regular epi, $C$ is a quotient of $U^{\mathbf{V}} F^{\mathbf{V}} A_C$ and so is in $\mathbf{V}$. Theorem 2.2 ensures that $\mathbf{V} \subseteq S^\perp$, and so $\mathbf{V} = S^\perp$.

It is easy to show that for collection $S$ of regular epis with projective domains, $S^\perp$ is closed under quotients. $\square$

Birkhoff's variety theorem in the classical setting is easily seen to be a direct corollary of Theorem 2.4.

## 3. The "Co-Birkhoff" Theorem

Dualizing the previous definitions, we have the following.

**Definition 3.1.** A category is a *quasi-co-Birkhoff category* if it is regularly well-powered, cocomplete and has epi-regular mono factorizations. If, in addition, the category has enough injectives, then it is a *co-Birkhoff category.*

A full subcategory of a quasi-co-Birkhoff category is a *quasi-covariety* iff it is closed under coproducts and codomains of epis. A quasi-covariety of a co-Birkhoff category is a *covariety* iff it is also closed under regular subobjects.

The property that a category *has enough injectives* is the dual of having enough projectives – i.e., every object is a regular subobject of an injective object (where injective objects, for our purposes, are injective *for regular monos*).

*Example* 3.1. Any Grothendieck topos is co-Birkhoff.

In order to consider covarieties of coalgebras, one would like to know when a category of coalgebras is co-Birkhoff. The following theorem gives a sufficient condition (which can be dualized easily for the algebraic case).

**Theorem 3.1.** *Let $\mathcal{E}$ be co-Birkhoff and $\Gamma : \mathcal{E} \longrightarrow \mathcal{E}$ preserve regular monos. Further, suppose that $U : \mathcal{E}_\Gamma \longrightarrow \mathcal{E}$ has a right adjoint, $H : \mathcal{E} \longrightarrow \mathcal{E}_\Gamma$ (so that $U$ is comonadic). Then the category $\mathcal{E}_\Gamma$ of $\Gamma$-coalgebras is also co-Birkhoff.*

*Proof.* It is well known that the coalgebraic forgetful functor has a right adjoint iff it is comonadic. See, for instance, [Tur96] for details.

One shows directly that $U$ reflects regular monos and so inherits the epi-regular mono factorizations from $\mathcal{E}$. This implies that $\mathcal{E}_\Gamma$ is regularly well-powered. Because $HA$ is injective if $A$ is injective, and $U$ reflects regular monos, $\mathcal{E}_\Gamma$ has enough injectives. □

Hereafter, we assume the conditions of Theorem 3.1 – namely, that $\mathcal{E}$ is co-Birkhoff, $\Gamma$ preserves regular monos and $U$ is comonadic (equivalently, has a right adjoint). It can easily be shown that, under these assumptions, $U$ preserves and reflects epis, regular monos and coproducts. In other words, $U$ creates the defining structure for covarieties.

As a consequence of Theorem 3.1, the dual of Theorem 2.4 applies to $\mathcal{E}_\Gamma$. Namely, a collection $\mathbf{V}$ of coalgebras is a covariety iff $\mathbf{V} = S_\perp$ for some collection $S$ of regular monos with injective codomain, where $S_\perp$ denotes the collection of coalgebras $\langle A, \alpha\rangle$ such that $\langle A, \alpha\rangle \perp i$ for all $i \in S$. From this result, we obtain the following natural definition of "coequation" (equivalent to those found in [Rut96] and [GS98]). A *coequation*[3] over an object $C$ of "colors" is a regular subobject of the cofree coalgebra $HC$ (see, for instance, [Rut96] for an overview of cofree coalgebras). A coalgebra $\langle A, \alpha\rangle$ satisfies a coequation $i : \langle E, \varepsilon\rangle \rightarrowtail HC$ just in case $\langle A, \alpha\rangle$ is *co-orthogonal* to $i$,

$$\langle A, \alpha\rangle \models i \Leftrightarrow \langle A, \alpha\rangle \perp i.$$

Explicitly, for every "coloring" $f : A \longrightarrow C$, the unique homomorphism

$$\tilde{f} : \langle A, \alpha\rangle \longrightarrow HC$$

factors (necessarily uniquely) through $i$, as shown below.

$$A \xrightarrow{f} C$$

$$\begin{tikzcd} \langle A, \alpha\rangle \arrow[r, "\tilde{f}"] \arrow[dr, dotted] & HC \\ & \langle E, \varepsilon\rangle \arrow[u, tail, "i"'] \end{tikzcd}$$

In yet other terms, $\langle A, \alpha\rangle \models i$ just in case every generalized element of $HC$ based at $\langle A, \alpha\rangle$ is in the subobject $i : \langle E, \varepsilon\rangle \rightarrowtail HC$. In this sense, a coequation is a "predicate".

Each coequation $i : \langle E, \varepsilon\rangle \longrightarrow HC$ determines a covariety, denoted $\{i\}_\perp$, of all coalgebras satisfying it. Also, any collection of coequations $S$ (allowing the cofree codomain to vary) determines a covariety, namely $S_\perp$. Call a full subcategory $\mathbf{V}$ of $\mathcal{E}_\Gamma$ a *coequational covariety* just in case $\mathbf{V} = S_\perp$ for some collection $S$ of regular subcoalgebras of cofree coalgebras. Then the dual of Theorem 2.4 immediately yields:

[3]In the algebraic case, there is a clear distinction between a single equation and a set of equations. We have found no simple distinction in the dual situation.

**Theorem 3.2.** *Let* $\mathbf{V}$ *be a full subcategory of* $\mathcal{E}_\Gamma$. *Then* $\mathbf{V}$ *is a Birkhoff covariety iff* $\mathbf{V}$ *is a coequational covariety.*

In particular, if we take $\mathcal{E}$ to be **Set** and $\Gamma$ to be a $\kappa$-bounded functor, as defined in [GS98], then each covariety is of the form $\{i\}_\perp$ for $i$ a regular mono into $H\kappa$, as was shown in ibid.

## 4. Behavioral Covarieties and Monochromatic Coequations

In typical applications of coalgebras in computer science, one is concerned with behavior "up to bisimulation". That is, if two coalgebras behave the same (according to bisimulation equivalence), then one does not distinguish the two, regardless of differences in "internal structure". Thus, one is often concerned with covarieties which are closed under total bisimulations, leading to the following definitions.

For present purposes, a *bisimulation* may be taken to be a relation in the category $\mathcal{E}_\Gamma$. That is, a bisimulation over $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ is a triple $\langle\langle R, \rho\rangle, r_1, r_2\rangle$ such that the maps

$$r_1 : \langle R, \rho\rangle \longrightarrow \langle A, \alpha\rangle\,,$$
$$r_2 : \langle R, \rho\rangle \longrightarrow \langle B, \beta\rangle$$

are jointly monic. For a more traditional definition suitable for coalgebras over co-Birkhoff categories, see [Hug01].

In [GS98], a *complete covariety* is defined as one closed under total bisimulations; we adopt the term *behavioral covariety* instead. A *total bisimulation* is one such that each projection is epi (in $\mathcal{E}_\Gamma$). A covariety $\mathbf{V}$ is *closed under total bisimulations* if, whenever $\langle A, \alpha\rangle \in \mathbf{V}$ and there is a total bisimulation relating $\langle A, \alpha\rangle$ to $\langle B, \beta\rangle$, then $\langle B, \beta\rangle$ is also in $\mathbf{V}$. It is shown, in ibid., that behavioral covarieties over **Set** are definable by coequations over 1. We generalize this result to our setting.

The following shows that the behavioral covarieties are exactly the covarieties which are *sinks*, in the terminology of Grigore Roşu ([Roş00]).

**Theorem 4.1.** *Let* $\mathcal{E}$ *be co-Birkhoff and* $\Gamma$ *preserve regular monos. Let* $\mathbf{V}$ *be a covariety of* $\mathcal{E}_\Gamma$. *The following are equivalent.*

1. $\mathbf{V}$ *is closed under total bisimulations.*

2. *If* $\langle A, \alpha\rangle \in \mathbf{V}$ *and there is a bisimulation* $\langle R, \rho\rangle$ *on* $\langle A, \alpha\rangle$ *and* $\langle B, \beta\rangle$ *such that the projection to* $\langle B, \beta\rangle$ *is epi, then* $\langle B, \beta\rangle$ *is in* $\mathbf{V}$.
3. $\mathbf{V}$ *is closed under domains of epis.*
4. $\mathbf{V}$ *is closed under domains of homomorphisms.*

*Proof.* **1⇔2:** Let $\langle A, \alpha\rangle \in \mathbf{V}$ and $\langle R, \rho\rangle$ be given as in 2, with projections $r_1$, $r_2$. Then $\mathsf{Im}(r_1) \in \mathbf{V}$ (as a regular subcoalgebra of $\langle A, \alpha\rangle$) and $\langle R, \rho\rangle$ is a total bisimulation between $\mathsf{Im}(r_1)$ and $\langle B, \beta\rangle$.

**2 ⇒ 3:** The graph of epis are total bisimulations.

**3⇒4:** Let $f:\langle A, \alpha\rangle \longrightarrow \langle B, \beta\rangle$ be given, $\langle B, \beta\rangle \in \mathbf{V}$, and take the epi-regular mono factorization, $f = i \circ p$. The domain of $i$ is in $\mathbf{V}$ as a regular subcoalgebra of $\langle B, \beta\rangle$. Hence $\langle A, \alpha\rangle \in \mathbf{V}$.

**4⇒ 1:** Let $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ be given and let $\langle R, \rho\rangle$ be a total bisimulation on $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$. Suppose, further, that $\langle A, \alpha\rangle \in \mathbf{V}$. Then, $\langle R, \rho\rangle \in \mathbf{V}$, since it is the domain of the projection

$$\langle R, \rho\rangle \longrightarrow \langle A, \alpha\rangle .$$

Since $\mathbf{V}$ is closed under codomains of epi homomorphisms, $\langle B, \beta\rangle \in \mathbf{V}$.

□

The following theorem is a generalization of ([GS98, Proposition 4.6]) and offers one last equivalent definition of behavioral covariety.

**Theorem 4.2.** *Let* $\mathcal{E}$ *and* $\Gamma$ *be as in Theorem 4.1. A full subcategory* $\mathbf{V}$ *of* $\mathcal{E}_\Gamma$ *is a behavioral covariety iff it is definable by a coequation over one color (i.e., by a regular subcoalgebra of the final coalgebra,* $H1$*).*

*Proof.* Let $\mathbf{V}$ be a behavioral covariety and let $U^{\mathbf{V}} \dashv H^{\mathbf{V}}$ be the associated regular mono coreflection, with unit $\varepsilon^{\mathbf{V}}$. We will show that $\mathbf{V} = \{\varepsilon^{\mathbf{V}}_{H1}\}_\perp$. Since "$\subseteq$" is clear, it suffices to show $\mathbf{V} \supseteq \{\varepsilon^{\mathbf{V}}_{H1}\}_\perp$.

Let $\langle A, \alpha\rangle$ be given and suppose that $\langle A, \alpha\rangle \perp \varepsilon^{\mathbf{V}}_{H1}$. Then $!:\langle A, \alpha\rangle \longrightarrow H1$ factors through $\varepsilon^{\mathbf{V}}_{H1}$, and so $\langle A, \alpha\rangle$ is the domain of an arrow into $U^{\mathbf{V}}H^{\mathbf{V}}H1$, which is in $\mathbf{V}$.

For the converse, let $\mathbf{V} = \{i\}_\perp$, where $i$ is a regular mono into $H1$. Let $p:\langle A, \alpha\rangle \twoheadrightarrow \langle B, \beta\rangle$ be given and suppose $\langle B, \beta\rangle \in \mathbf{V}$. Then $!_\beta:\langle B, \beta\rangle \longrightarrow H1$

factors through $i$, say, $!_\beta = i \circ f$. Consequently, $!_\alpha = i \circ f \circ p$. Since $!_\alpha$ is the only map from $\langle A, \alpha \rangle$ to $H1$, it follows that $\langle A, \alpha \rangle \perp i$. $\square$

It is instructive to compare this theorem to its dual, which says that a variety of algebras is closed under codomains of monos iff it is definable by a set of equations with no variables. Since variable-free equations identify terms built up out of constants (only), it is clear that, if a subalgebra satisfies a variable free-equation, so does the algebra[4] (because all of the constants for the signature must be interpreted in the subalgebra). We don't know of any simpler way of seeing the converse than chasing through the proof above, reversing the arrows.

One can also consider a covariety closure operation, taking a covariety to the least behavioral covariety containing it. Specifically, we define an operator

$$\mathrm{CoVar}(\mathcal{E}_\Gamma) \longrightarrow \mathrm{CoVar}(\mathcal{E}_\Gamma)$$

taking a covariety $\mathbf{V}$ to the collection $\overline{\mathbf{V}}$, where $\langle A, \alpha \rangle \in \overline{\mathbf{V}}$ iff there is some map $f : \langle A, \alpha \rangle \longrightarrow \langle B, \beta \rangle$ with $\langle B, \beta \rangle \in \mathbf{V}$.

It is easy to show that this closure produces another covariety. Hence,

**Theorem 4.3.** *If $\mathbf{V}$ is a covariety, then $\overline{\mathbf{V}}$ is a behavioral covariety.*

The next theorem states in coequational terms how to obtain $\overline{\mathbf{V}}$. We know that $\mathbf{V}$ is defined by a collection of coequations, in the sense that $\mathbf{V}$ is exactly the class of coalgebras co-orthogonal to a collection of regular monos with cofree codomains. In fact, we can say more about the collection of regular monos – namely, that they are the components of the counit of a regular mono co-reflection (Corollary 2.3), we show that this counit also gives a defining coequation for $\overline{\mathbf{V}}$. Of course, since $\overline{\mathbf{V}}$ is a behavioral covariety, the only cofree coalgebra one needs to consider is the final coalgebra.

**Theorem 4.4.** *Let $\mathbf{V}$ be a covariety and $\varepsilon^{\mathbf{V}} : U^{\mathbf{V}} H^{\mathbf{V}} \longrightarrow 1_{\mathcal{E}_\Gamma}$ be the counit of the associated adjunction*

$$\begin{tikzcd} \mathbf{V} \arrow[r, bend left, "U^{\mathbf{V}}"] \arrow[r, phantom, "\perp"] & \mathcal{E}_\Gamma \arrow[l, bend left, "H^{\mathbf{V}}"] \end{tikzcd}$$

*Then $\overline{\mathbf{V}} = \{\varepsilon^{\mathbf{V}}_{H1}\}_\perp$.*

[4]Moreover, if $f : \langle A, \alpha \rangle \longrightarrow \langle B, \beta \rangle$ is any algebra homomorphism and $\langle A, \alpha \rangle \models E$, where $E$ is a set of variable-free equations, then $\langle B, \beta \rangle \models E$.

*Proof.* Let $\langle A, \alpha\rangle \in \overline{\mathbf{V}}$. Then there is an $f : \langle A, \alpha\rangle \longrightarrow \langle B, \beta\rangle$ such that $\langle B, \beta\rangle \in \mathbf{V}$. Since $\langle B, \beta\rangle \in \mathbf{V}$, $\langle B, \beta\rangle \perp \varepsilon^{\mathbf{V}}_{H1}$. Consequently, $\langle A, \alpha\rangle \perp \varepsilon^{\mathbf{V}}_{H1}$.

On the other hand, if $\langle A, \alpha\rangle \perp \varepsilon^{\mathbf{V}}_{H1}$, then the factorization of $\langle A, \alpha\rangle \longrightarrow H1$ through $\varepsilon^{\mathbf{V}}_{H1}$ is a homomorphism into a coalgebra in $\mathbf{V}$. Hence $\langle A, \alpha\rangle \in \overline{\mathbf{V}}$. $\square$

Note that behavioral covarieties are defined by a single coequation, regardless of any boundedness conditions on $\Gamma$.

4.1. **An example of covariety which is not behavioral.** As the work of the preceding section indicates, behavioral covarieties are relatively well-understood classes of coalgebras. Inasmuch as coalgebras are distinguished in computer science only "up to behavioral equivalence", one may ask whether non-behavioral covarieties offer any particular interest. We offer here an example of a covariety which is not behavioral, but which can be described by a coequation over two colors. This covariety arises in a natural way and, we hope, gives some indication of the added expressive power that "multi-colored" coequations offer.

Consider the functors $\mathbb{N} \times -$ and $1 + \mathbb{N} \times -$ on the category $\mathbf{Set}$. As usual, we think of coalgebras for these functors as collections of streams over $\mathbb{N}$ (see [JR97], for instance). In particular, a coalgebra for $\mathbb{N} \times -$ can be thought of as a collection of infinite streams, closed under the tail destructor. A coalgebra for $1 + \mathbb{N} \times -$ can be understood as a collection of finite or infinite streams over $\mathbb{N}$, again closed under the tail destructor (when defined).

It is clear that the category $\mathbf{Set}_{\mathbb{N}\times -}$ is a full subcategory of $\mathbf{Set}_{1+\mathbb{N}\times -}$. What is less obvious is that one can regard $\mathbf{Set}_{1+\mathbb{N}\times -}$ as a full subcategory of $\mathbf{Set}_{\mathbb{N}\times -}$, and it is this perspective on which we will focus. Define a functor

$$I : \mathbf{Set}_{1+\mathbb{N}\times -} \longrightarrow \mathbf{Set}_{\mathbb{N}\times -}$$

as follows. If $\langle A, \alpha\rangle$ is a $1 + \mathbb{N} \times -$ coalgebra, then $I(\langle A, \alpha\rangle) = \langle A, \alpha\rangle'$ will be a $\mathbb{N} \times -$ coalgebra. Specifically, let $\alpha'$ be defined by

$$\alpha'(a) = \begin{cases} \langle 0, a\rangle & \text{if } \alpha(a) = * \\ \langle h_\alpha(a) + 1, t_\alpha(a)\rangle & \text{else} \end{cases}$$

Intuitively, $I$ takes infinite lists to the list one gets by applying successor in each position. For finite lists, $I$ again applies successor in each position and then tacks on 0's at the end. However, the 0's are tacked on in a particular manner – once we hit 0 in the list, the "state" never changes. We stay at the same element of $A$

and continue outputting 0's. This description should lend plausibility to the claim that $\mathbf{V}$ is not behavioral, which we will later prove. The property that a coalgebra stabilizes at a particular state is not a property closed under total bisimulation.

It is routine to check that $I$ defines a functor and, furthermore, that $I$ is full, faithful and injective on objects. Let $\mathbf{V}$ be the image of $\mathbf{Set}_{1+\mathbb{N}\times -}$ under $I$. One could check directly that $\mathbf{V}$ is a covariety, but we prefer to explicitly give a defining coequation instead.

Let $\Omega = \{\top, \bot\}$. We will define a pair of maps

$$H2 \rightrightarrows H\Omega$$

such that their equalizer is a defining coequation for $\mathbf{V}$. Since defining a map $H2 \longrightarrow H\Omega$ is the same as defining a map $UH2 \longrightarrow \Omega$, we do that instead[5]. Let $\langle h, t\rangle$ be the structure map on $UH2$ and let $\varepsilon$ be the counit of the adjunction $U \dashv H$. For each $\sigma \in H2$, let

$$\mu(\sigma) = \begin{cases} \top & \text{if } h(\sigma) \neq 0 \\ \top & \text{if } \varepsilon_2(\sigma) = \varepsilon_2 \circ t(\sigma) \\ \bot & \text{else} \end{cases}$$

Let $\mathsf{true}\colon UH2 \longrightarrow \Omega$ be the map taking each $\sigma$ to $\top$. Let $i\colon\langle E, \varepsilon\rangle \longrightarrow H2$ be the equalizer of the adjoint transposes $\widetilde{\mathsf{true}}$ and $\widetilde{\mu}$. We will show that $\mathbf{V} = \{i\}_\bot$.

Suppose that $\langle A, \alpha\rangle \in \mathbf{V}$. We would like to show that $\langle A, \alpha\rangle \perp i$. We will write $h_\alpha$ and $t_\alpha$ for the compositions $\pi_1 \circ \alpha$ and $\pi_2 \circ \alpha$, respectively. We have, then, for every $a \in A$, if $h(a) = 0$, then $t(a) = a$. Let $c\colon A \longrightarrow 2$ be a coloring of $A$. If, for each $a \in A$, $\mu \circ U\widetilde{c}(a) = \top$, then it follows that $\langle A, \alpha\rangle \perp i$, as desired. This follows readily from the definition of $\mu$.

On the other hand, suppose that $\langle A, \alpha\rangle \perp i$. We would like to show that $\langle A, \alpha\rangle \in \mathbf{V}$. It suffices to show that, for each $a \in A$, if $h_\alpha(a) = 0$, then $t_\alpha(a) = a$. So, suppose that $h_\alpha(a) = 0$ and define $c\colon A \longrightarrow 2$ by

$$c(b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{else} \end{cases}$$

Then, $\mu \circ U\widetilde{c}(a) = \top$, so it follows that $c \circ t(a) = c(a)$ and hence $t(a) = a$.

*Remark* 4.1. While this coequation defines the covariety $\mathbf{V}$, it is worth noting that $E$ is not itself in the covariety. Instead, there is a proper regular subcoalgebra $E'$

[5] This also corresponds to our intuition that coequations are "really" predicates on the carriers of cofree coalgebras

of $E$ which is in the covariety, and whose inclusion is also a defining coequation for $\mathbf{V}$. This coequation is given by altering the definition of $\mu$ above, so that

$$\mu'(\sigma) = \begin{cases} \top & \text{if } h(\sigma) \neq 0 \\ \top & \text{if } \varepsilon_2(\sigma) = \varepsilon_2 \circ t(\sigma) \text{ and } h \circ t(\sigma) = 0 \\ \perp & \text{else} \end{cases}$$

We then take $E'$ to be the equalizer of the adjoint transposes of $\mu'$ and $\mathsf{true}$, as before.

Algebraically, this situation occurs as well. Let $E$ be a set of equations over $X$ and take the quotient $FX/E$. Then, $FX/E$ need not be in the variety defined by $E$. In fact, $FX/E$ satisfies $E$ iff $E$ is deductively complete, in the sense of containing all of the equations deducible from those in $E$.

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Department of Philosophy, Carnegie Mellon University
*E-mail address*: awodey@cmu.edu, jesse@cmu.edu