A Study of Categories of Algebras and Coalgebras

Jesse Hughes

May, 2001

Department of Philosophy
Carnegie Mellon University
Pittsburgh PA 15213

Thesis Committee
Steve Awodey, Co-Chair
Dana Scott, Co-Chair
Jeremy Avigad
Lawrence Moss, Indiana University

Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy

Abstract

This thesis is intended to help develop the theory of coalgebras by, first, taking classic theorems in the theory of universal algebras and dualizing them and, second, developing an internal logic for categories of coalgebras.

We begin with an introduction to the categorical approach to algebras and the dual notion of coalgebras. Following this, we discuss (co)algebras for a (co)monad and develop a theory of regular subcoalgebras which will be used in the internal logic. We also prove that categories of coalgebras are complete, under reasonably weak conditions, and simultaneously prove the well-known dual result for categories of algebras. We close the second chapter with a discussion of bisimulations in which we introduce a weaker notion of bisimulation than is current in the literature, but which is well-behaved and reduces to the standard definition under the assumption of choice.

The third chapter is a detailed look at three theorem's of G. Birkhoff [**Bir35, Bir44**], presenting categorical proofs of the theorems which generalize the classical results and which can be easily dualized to apply to categories of coalgebras. The theorems of interest are the variety theorem, the equational completeness theorem and the subdirect product representation theorem. The duals of each of these theorems is discussed in detail, and the dual notion of "coequation" is introduced and several examples given.

In the final chapter, we show that first order logic can be interpreted in categories of coalgebras and introduce two modal operators to first order logic to allow reasoning about "endomorphism-invariant" coequations and bisimulations internally. We also develop a translation of terms and formulas into the internal language of the base category, which preserves and reflects truth. Lastly, we introduce a Kripke-Joyal style semantics for $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$, as well as a pointwise semantics which reflects the intuition of coequation forcing at a point or subset of a coalgebra.

Acknowledgments

I have been fortunate to have two advisors on this dissertation. I first became interested in the subject thanks to Dana Scott, who helped guide the questions and suggested the Birkhoff's theorem research in particular. Steve Awodey taught me everything I know about category theory, but I am grateful anyway. Both advisors helped my writing immensely, in addition to guiding my research, and I am thankful for their patience and wisdom.

When Dana first suggested I look into coalgebras, he pointed me to *Vicious Circles*, by Jon Barwise and Larry Moss. Since that book was the start of my study of coalgebras, it seemed only fair that Larry Moss should have to read this dissertation. He graciously agreed to be my outside reader. I am grateful for the advice he and Jeremy Avigad gave as members of my committee.

My research has benefited through discussions and correspondence with many people, including Peter Aczel, Jiří Adámek, Andrej Bauer, Lars Birkedal, Steve Brookes, Corina Cîrstea, Federico do Marchi, Neil Ghani, Jeremy Gibbons, Peter Gumm, Bart Jacobs, Alexander Kurz, Bill Lawvere, John Reynolds, Tobias Schröder James Worrell and Jaap van Oosten and others I'm sure to have missed here. I also want to thank the organizers of the Coalgebraic Methods for Computer Science workshop for providing a great opportunity to meet and discuss our research.

On a more personal note, I could not have completed this work without the extraordinary patience and generosity of my wife, Ling Cheung. In fact, I am the rare husband who's also grateful for the extended visits of his mother-in-law, Siu Kai Lam. She helped out considerably when two graduate students were overwhelmed with a newborn, Quincy Prescott Hughes. I also enjoyed the distractions from my work, including Penguins hockey, regular fishing trips with Dirk Schlimm, exciting demolition derbies at New Alexandria and captivating and suspenseful games of Peek-a-boo with Quincy.

Contents

Introduction 1

Chapter synopsis 4

Chapter 1. Algebras and coalgebras 7

1.1. Algebras and coalgebras for an endofunctor 7

1.2. Structural features of $\mathcal{E}^\Gamma$ and $\mathcal{E}_\Gamma$ 16

1.3. Subalgebras 22

1.4. Congruences 27

1.5. Initial algebras and final coalgebras 32

Chapter 2. Constructions arising from a (co)monad 47

2.1. (Co)monads and (co)algebras 47

2.2. Subcoalgebras 61

2.3. Subcoalgebras generated by a subobject 73

2.4. Limits in categories of coalgebras revisited 77

2.5. Bisimulations 89

2.6. Coinduction and bisimulations 105

2.7. n-simulations 109

Chapter 3. Birkhoff's variety theorem 113

3.1. The classical theorem 114

3.2. A categorical approach 115

3.3. Categories of algebras 125

3.4. Uniformly Birkhoff categories 127

3.5. Deductive closure 135

3.6. The coalgebraic dual of Birkhoff's variety theorem 140

3.7. Uniformly co-Birkhoff categories 153

3.8. Invariant coequations 162

3.9. Behavioral covarieties and monochromatic coequations 168

Chapter 4. The internal logic of $\mathcal{E}_\mathbb{G}$ 175

4.1. Preliminary results 175

4.2. Transfer principles 186

4.3. A Kripke-Joyal style semantics 198
4.4. Pointwise forcing of coequations 200

Concluding remarks and further research 205

Appendix A. Preliminaries 209
A.1. Notation 209
A.2. Factorization systems 209
A.3. Predicates and Subobjects 212
A.4. Relations 214
A.5. Monads and comonads 216

Appendix. Bibliography 219

Appendix. Index 223

Introduction

The theory of universal algebras has been well-developed in the twentieth century. The theory has also proved especially fruitful, with early results (like Birkhoff's variety theorem) providing a basis for model theory and other results providing an abstract understanding of familiar principles of induction, recursion and freeness. The theory of coalgebras is considerably younger and less well developed. Coalgebras arise naturally, as Kripke models for modal logic, as automata and objects for object oriented programming languages in computer science, etc. Hence, one would like a unified theory of coalgebras to play a role analogous to that of the theory of algebras. This goal is aided by the duality between algebras and coalgebras. Statements about categories of algebras yield dual statements about categories of coalgebras. One can then investigate whether there are reasonable assumptions about the categories of coalgebras that yield the dual theorems.

Algebras, in their commonest form, can be understood as a set together with some operations on the set. In other words, algebras are structures for a signature. The term algebras are examples of free algebras, where freeness is easily expressed in terms of adjoint functors. Such free algebras (which are initial objects in a related category of algebras) come with the proof principle of induction, which can be understood in terms of minimality. That is, the principle of induction is equivalent to the property that an algebra has no non-trivial subalgebras. The property of definition by recursion is exactly the property that an algebra is an initial object. Thus, these familiar topics of universal algebra are well-suited for a categorical setting. We can use the tools of category theory to investigate freeness, induction and recursion as special cases of adjointness, minimality and initiality, respectively. In particular, these algebraic properties can be represented as standard categorical properties applied to categories of algebras (in which the structure of the category leads to the well-known algebraic properties).

Coalgebras can also be regarded as a set together with certain operations on it, but with a key difference. Where an algebra is intended to model combinatorial operations, a coalgebra models a set with various unary operations whose codomain is a (typically) more complex structure. These operations can be viewed as "destructors" which take an element of the coalgebra to its constituent parts. Compare this view with the notion that an algebras operations give a means (not necessarily unique) of "constructing" an element out of a tuple.

Consider, for instance, a set S of A-labeled binary trees[1] which is closed under the "childOf" relation. That is, if $x \in S$, then both the left and right subtrees of x (if they exist) are also in S. Then S has a natural coalgebraic structure consisting of three destructor functions. Given any $x \in S$, we may ask for the label of x. We may also ask for the left child or right child of x, assuming that there is an "error state" which can be returned if x has no such child . These three structure maps define a signature Σ for a category of coalgebras in the same way that a set with some combinatorial operations define a signature for a category of algebras (i.e., a similarity type). Any set X, together with three operations,

$$a : X \longrightarrow A,$$
$$l : X \longrightarrow X + 1,$$
$$r : X \longrightarrow X + 1,$$

is a Σ-coalgebra. Equivalently, any set X with a single map

$$\langle a, l, r\rangle : X \longrightarrow A \times (X + 1) \times (X + 1)$$

is a coalgebra of the same type as our set S of binary trees. Indeed, any such structured set can be regarded as a set of trees itself.

We can use the theory of algebras in order to develop the theory of coalgebras. The duality is apparent in the distinguished initial algebra/final coalgebra. The initial algebra is the initial (i.e., "least") fixed point of the associated functor, while the final coalgebra is the final (i.e., "greatest") fixed point. The initial algebra comes equipped with principles of recursion and induction, while the final coalgebra satisfies the principles of corecursion and coinduction, that is, principles which are appropriate to collections of non-well-founded structures. Intuitively, the elements of the initial algebra are those which can be constructed from some set of basic elements in a finite number of steps, while the elements of the final coalgebra are all of those structures of the appropriate signature, including those for which no finite construction is apparent (think of the distinction between well-founded binary trees and non-well-founded binary trees). Of course, the extent to which this intuition is appropriate depends on the functor (i.e., signature) at hand. But the point of this comparison remains: To construct a theory of coalgebras, one may take the theory of algebras and dualize the central theorems. One then interprets the result in order to make sense of it – the traditional statement of the principle of coinduction, for instance, does not make apparent its duality with induction. Similarly, the description of a cofree coalgebra

[1] In this example, we do not require that a tree have both a left and a right child if it has any children.

bears little superficial analogy to the corresponding view that free algebras are term algebras. Instead, some work was required to give a useful description of cofreeness, apart from the categorical task of "turning the arrows around".

The categorical task can be non-trivial as well. Classical results in universal algebra theory were proved in a fairly narrow (from a categorical perspective) setting. In order to dualize these classical theorems, one must first translate the proof into categorical terms, in order to see which properties of **Set** and polynomial functors are relevant to the theorem. Then, one may dualize these properties – and hope that the result yields reasonable assumptions for the category of coalgebras! If not, then a bit more work may be required to ensure that the proof goes through.

This method has special difficulties when the algebraic proof intrinsically involves elements of algebras. Unfortunately, the dual of "global elements" yields nothing worthwhile and one must find other means of proving the dual theorem. This problem can be seen in the proof of Birkhoff's "co-subdirect product" theorem in Section 3.7.1. The proof of this theorem bears no real resemblance to the proof of its algebraic counterparts. Furthermore, the statement of the theorem required assumptions beyond those in the original theorem. These differences reflect the difficulty of dualizing a theorem whose proof involves reasoning about elements of algebras.

This thesis is largely an extended exercise in the program of dualizing algebraic results in order to understand categories of coalgebras. The main result in this direction is the dual of Birkhoff's variety theorem, which we treat in considerable detail in Chapter 3. In addition, we consider his deductive completeness theorem and dualize this theorem, yielding a modal operator on categories of coalgebras which is the dual of closing sets of equations under deductive consequence, and his subdirect product theorem.

One may hope, as well, that as the theory of coalgebras matures, developments in the theory may lead to corresponding results for algebras. This thesis features two modest steps in that direction. First, the modal operator for bisimulations dualizes to a closure operator on relations over coproducts of algebras – but it's unclear what applications this closure might have. Second, in Section 3.9.3, we consider classes of algebras defined by equations with no variables (just constants) and show that these are exactly the varieties closed under codomains of homomorphisms. This theorem may be well-known (although a search turned up nothing), but illustrates the way in which a coalgebraic topic (covarieties closed under bisimulation) can, when dualized, yield natural algebraic results.

Birkhoff's variety and completeness theorems are fundamental to the theory of algebras, establishing equational reasoning as the "right" logic for algebras. Hence, it is natural to suppose that "coequations" will play an important role in understanding categories of coalgebras. The work in proving the "co-Birkhoff" theorems yields a definition of coequation which is easily understood: A coequation over C is a predicate on the cofree coalgebra over C (where "cofreeness" now requires some explanation, of course!). Then, to the extent that coequations are central to reasoning about coalgebras, one can infer that the "right" logic for coalgebras is a predicate (not "equational") logic.

This inference helps motivate the final chapter, in which we develop a logic which can be interpreted in categories of coalgebras (i.e., an "internal" logic). In addition to the first order core of the logic, we introduce a modal operator arising from the dual of Birkhoff's completeness theorem. Furthermore, we make use of a translation of statements in the logic of the category $\mathcal{E}_{\mathbb{G}}$ of coalgebras to the base category $\mathcal{E}$. This translation allows "transition" rules which take as premises statements in $\mathcal{L}(\mathcal{E})$ and form conclusions in $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$ (and *vice versa*). We also give a Kripke-Joyal style semantics which arises naturally from pointwise satisfaction of equations.

Throughout, we work to develop results which apply to as broad a setting as possible. While most research in categories of coalgebras take the base category **Set** as the starting point (and perhaps even limit discussion to an inductively specified set of functors), we work to develop results which apply to a wide number of categories and functors. One topic in which the difference is most apparent is the notion of bisimulation. Because we do not assume choice, the traditional notion of bisimulation is too restrictive – two elements which are behaviorally indistinguishable need not be "bisimilar" under that definition. Consequently, we offer a new definition of bisimulation in Section 2.5. We show that the new definition reduces to the traditional definition under the axiom of choice. Regardless of the axiom of choice, the new definition is reasonably well-behaved (although without choice or preservation of pullbacks, it's not clear the bisimulations compose), which cannot be said for the old definition.

In summary, then, this thesis has three primary goals. First, help develop a theory of coalgebras by dualizing results in algebra theory and, when appropriate, dualizing new coalgebraic results and interpret them as theorems about algebras. Second, develop an internal (modal) logic for categories of coalgebras in which coequations play a central role and in which there is an interplay between derivations in the base category and derivations in the category of coalgebras. Third, do the above in as general a setting as practicable, modifying previous definitions, if necessary, to be suitable for the general setting (always ensuring that they reduce to familiar definitions in the familiar setting).

Chapter synopsis

Chapter 1: In this chapter, we introduce the categorical definitions of algebra and coalgebra. We discuss some basic structural features of the category of algebras, $\mathcal{E}^\Gamma$, and the category of coalgebras, $\mathcal{E}_\Gamma$. We spend some time discussing subalgebras, congruences and exactness properties in $\mathcal{E}^\Gamma$ as an exercise in applying categorical reasoning to this generalization of universal algebras. Finally, we discuss the initial algebra and final coalgebra. Each of these come equipped with certain proof principles. The initial algebra satisfies the proof principles of induction and definition by recursion, while the final coalgebra satisfies the dual principles of coinduction and definition by corecursion. We highlight the duality when presenting these principles.

Chapter 2: We discuss the relationship between algebras for a monad and free algebras for an endofunctor and the dual result involving coalgebras for a comonad and cofree coalgebras. Following this, we introduce subcoalgebras and discuss a left and a right adjoint to the subcoalgebraic forgetful functor. We use the right adjoint to prove that, in the presence of cofree coalgebras, the category $\mathcal{E}_\Gamma$ is as complete as $\mathcal{E}$. The presence of products in $\mathcal{E}_\Gamma$ leads to a discussion of relations over coalgebras. In Section 2.5, we introduce a new definition of bisimulation – one which is appropriate to coalgebras in categories without the axiom of choice. We close with a discussion of the relation between coinduction and bisimulations.

Chapter 3: In this section, we primarily discuss Birkhoff's variety theorem [**Bir35**] and its dual. To begin, we discuss a generalization of equation satisfaction that is more suitable for a categorical analysis – namely, orthogonality conditions. This leads to an abstract proof of Birkhoff's theorem which applies to a wide range of categories, and in particular applies to certain categories of algebras. This approach naturally dualizes to provide the "co-Birkhoff" theorem for covarieties of coalgebras. In addition, we consider Birkhoff's deductive completeness theorem, ibid, and show how its dual leads to a natural modal operator on coalgebraic predicates. In addition, we discuss the dual of Birkhoff's subdirect product theorem, extending the work in [**GS98**].

Chapter 4: We show that, given some reasonably weak assumptions on $\mathcal{E}$ and Γ, the category $\mathcal{E}_\Gamma$ can interpret first order logic. We provide a translation from the internal language of $\mathcal{E}_\Gamma$ to the internal language of $\mathcal{E}$ which preserves entailment. This translation explicitly involves augmenting the language $\mathcal{E}$ with the modal operator $\Box$ from Chapter 2. We close with a brief discussion of Kripke-Joyal semantics and pointwise semantics which are suggested from the coequation-as-predicate viewpoint.

CHAPTER 1

Algebras and coalgebras

In this chapter, we present some preliminary definitions and results for categories of algebras and coalgebras. We begin by developing the theories side by side, using the natural dualities to derive results for coalgebras by dualizing results for algebras. In Section 1.2, we discuss limits and colimits in categories $\mathcal{E}^{\Gamma}$ and $\mathcal{E}_{\Gamma}$, focusing on those (co-)limits which are created by the respective forgetful functor. We also discuss factorizations in $\mathcal{E}^{\Gamma}$ and $\mathcal{E}_{\Gamma}$ which are inherited from the base category. In Section 1.3, we discuss subalgebras, postponing the dual notion until Chapter 2. Similarly, in Section 1.4, we present the standard (categorical) development of algebraic relations (i.e., pre-congruences), while postponing the introduction of coalgebraic relations and bisimulations until the following chapter, when we have already constructed products. We conclude with a discussion of initial algebras and final coalgebras and the characteristic properties (induction/recursion and coinduction/corecursion, respectively).

1.1. Algebras and coalgebras for an endofunctor

We start with the definitions of Γ-algebras and Γ-coalgebras for endofunctor Γ. Note that this is not the same definition as (co)algebras for a (co)monad, which we discuss in Chapter 1.1. Essentially, a category of (co)algebras for an endofunctor is equivalent to a category of (co)algebras for a (co)monad just in case there are (co)free (co)algebras for each object in the base category.

1.1.1. Definitions. We briefly state the definitions of Γ-algebra, Γ algebra-homomorphism and $\mathcal{E}^{\Gamma}$ and then dualize. The aim is that the reader, who is likely familiar with universal algebras in some form, should find the definition of coalgebra familiar and natural as the dual of an algebra. In Section 1.1.3, we will give some examples of coalgebras to show that coalgebras arise naturally.

Definition 1.1.1. Let $\mathcal{E}$ be any category. Given an endofunctor $\Gamma\colon\mathcal{E}\to\mathcal{E}$, a Γ-*algebra* consists of a pair $\langle A, \alpha\rangle$, where A is an object of $\mathcal{E}$ and $\alpha\colon\Gamma A\to A$ an arrow in $\mathcal{E}$. We call A the *carrier* and α the *structure map* of the algebra

Given two Γ-algebras, $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$, a Γ-*algebra homomorphism,*

$$f\colon\langle A, \alpha\rangle\longrightarrow\langle B, \beta\rangle,$$

is a map $f : A \to B$ in $\mathcal{E}$ such that the following diagram commutes.

$$\begin{tikzcd} \Gamma A \arrow[r, "\Gamma f"] \arrow[d, "\alpha"'] & \Gamma B \arrow[d, "\beta"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

The Γ-algebras and their homomorphisms form a category, denoted $\mathcal{E}^{\Gamma}$.

The concept of Γ-coalgebras is formally dual to the definition of Γ-algebra above. Specifically, the category $\mathcal{E}_{\Gamma}$ of coalgebras arises formally as the category $((\mathcal{E}^{\mathsf{op}})^{\Gamma^{\mathsf{op}}})^{\mathsf{op}}$. Of course, interest in coalgebras comes from the fact the these structure arise independently as well, from computer science semantics, Kripke frames and models, and other sources.

Definition 1.1.2. A Γ-*coalgebra* is a $\langle A, \alpha \rangle$, where $\alpha : A \to \Gamma A$. Again, A is the *carrier* and α the *structure map* of the coalgebra. A Γ-*coalgebra homomorphism* is again a commutative square:

$$\begin{tikzcd} \Gamma A \arrow[r, "\Gamma f"] & \Gamma B \\ A \arrow[r, "f"'] \arrow[u, "\alpha"] & B \arrow[u, "\beta"'] \end{tikzcd}$$

The Γ-coalgebras and their homomorphisms again form a category, denoted $\mathcal{E}_{\Gamma}$.

Note: We often refer to Γ-algebra homomorphisms as Γ-homomorphisms or just homomorphisms. We do the same for coalgebra homomorphisms. The kind of homomorphism we mean should be clear from the context.

For each of these categories, there is an evident forgetful functor, U, taking a (co)algebra $\langle A, \alpha \rangle$ to A. Properly, we should write

$$U^{\Gamma} : \mathcal{E}^{\Gamma} \longrightarrow \mathcal{E},$$
$$U_{\Gamma} : \mathcal{E}_{\Gamma} \longrightarrow \mathcal{E},$$

to indicate that these are different functors, depending on whether we are interested in algebras or coalgebras and also depending on the functor Γ. Of course, we will avoid such complications and the meaning of U should be clear from context.

In Section 1.2, we will give some of the features of the categories $\mathcal{E}^{\Gamma}$ and $\mathcal{E}_{\Gamma}$. In particular, the forgetful functor creates limits (colimits, resp.) in categories of algebras (coalgebras, resp.). Before exploring these features, we give some examples of categories of algebras and coalgebras.

REMARK 1.1.3. The notation for a Γ-algebra is the same as that for a Γ-coalgebra. Namely, each is a pair $\langle A, \alpha \rangle$, where

$$\alpha : \Gamma A \longrightarrow A$$

in the algebraic case, and

$$\alpha : A \longrightarrow \Gamma A$$

in the coalgebraic case. Most often, whether we mean $\langle A, \alpha \rangle$ to be an algebra or a coalgebra will be clear from context. However, we sometimes use this ambiguity of notation to our advantage. For example, in Section 1.3, we note that a subobject in $\mathcal{E}^{\Gamma}$ is a monic algebra homomorphism

$$\langle B, \beta \rangle \rightarrowtail \langle A, \alpha \rangle \ .$$

Also, a subobject in $\mathcal{E}_{\Gamma}$ is a monic coalgebra homomorphism

$$\langle B, \beta \rangle \rightarrowtail \langle A, \alpha \rangle \ .$$

Since the notation for each is the same, we can draw the diagram just once and say

A subobject of a Γ-(co)algebra is a monic homomorphism

$$\langle B, \beta \rangle \rightarrowtail \langle A, \alpha \rangle \ .$$

1.1.2. Some examples of algebras. In this section, we begin with some examples of algebras for various functors. We will, in each case, make clear what the homomorphisms in $\mathcal{E}^{\Gamma}$ are.

EXAMPLE 1.1.4. Consider the functor $\Gamma : \mathbf{Set} \to \mathbf{Set}$ given by

$$\Gamma A = 1 + A \times A.$$

An algebra for this functor consists of a set A together with a structure map

$$\alpha : A \times A + 1 \longrightarrow A.$$

Such a map α is equivalent to a pair of maps

$$\cdot_\alpha : A \times A \longrightarrow A, \text{ and}$$
$$a : 1 \longrightarrow A$$

In other words, a Γ-algebra is a triple $\langle A, \cdot_\alpha, a \rangle$, where $\cdot_\alpha$ is a binary operation on A and a is a distinguished element of A. This is also called a Σ-model or Σ-structure for the signature

$$\Sigma = \{ \cdot^{(2)}, e^{(0)} \}.$$

See Example 1.1.5 for details.

Given another Γ-algebra, $\langle B,\, \cdot_\beta,\, b\rangle$, a Γ-homomorphism $\langle A,\, \cdot_\alpha,\, a\rangle \longrightarrow \langle B,\, \cdot_\beta,\, b\rangle$ is a map $f\colon A \to B$ such that the following diagram commutes:

$$\begin{tikzcd}
A \times A + 1 \arrow[r, "f\times f+\mathsf{id}_1"] \arrow[d, "\langle \cdot_\alpha, a\rangle"'] & B \times B + 1 \arrow[d, "\langle \cdot_\beta, b\rangle"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

This entails that

$$\begin{aligned} f(s \cdot_\alpha t) &= f(s) \cdot_\beta f(t), \text{ and} \\ f(a) &= b. \end{aligned}$$

In other words, a homomorphism is a map that respects the binary operation and constant. The next example generalizes this result to arbitrary universal algebras.

Example 1.1.5. Much of this dissertation is devoted to taking well-known results in universal algebra, translating them to a categorical setting and dualizing. This approach relies on the fact that the categorical notion of algebra for an endofunctor is a proper generalization of the notion of universal algebra. In particular, given any signature Σ, there is a polynomial functor $\mathbb{P}$ such that the category $\mathbf{Set}^{\mathbb{P}}$ is the category of universal Σ-algebras. This result is well-known, but it is useful to work through the details here, in order to gain some familiarity with the categorical notions.

These definitions can be found in [**MT92**], [**Grä68**] and elsewhere.

A *signature* Σ is a set of function symbols together with associated (finite) arities. We write $f^{(n)}$ to indicate that f is a function symbol of arity n. If the arity of a function symbol c is 0, then we call $c^{(0)}$ a *constant symbol*.

A Σ-*algebra* is a pair

$$\mathcal{S} = \langle S,\, \{ f^{(n)}_{\mathcal{S}} \colon S^n \longrightarrow S \,|\, f^{(n)} \in \Sigma \} \rangle,$$

where S is a set (called *the carrier of the algebra*). Notice that the interpretation of a constant symbol is an element of S.

Given two Σ algebras $\mathcal{S}$ and $\mathcal{T}$, we say that a set function $\phi\colon S \to T$ is a Σ-homomorphism if, for every function symbol $f^{(n)}$ in Σ, and every $s_1, \ldots, s_n \in S$,

$$f^{(n)}_{\mathcal{T}}(\phi(s_1), \ldots, \phi(s_n)) = \phi(f^{(n)}_{\mathcal{S}}(s_1, \ldots, s_n)).$$

In particular, this means that for every constant symbol $c^{(0)}$, $\phi(c^{(0)}_{\mathcal{S}}) = c^{(0)}_{\mathcal{T}}$.

Given a signature Σ, consider the polynomial functor $\mathbb{P}\colon \mathbf{Set} \to \mathbf{Set}$ given by

$$\mathbb{P}S = \coprod_{f^{(n)} \in \Sigma} S^n.$$

Of course, if Σ is infinite, then this functor involves an infinite coproduct, so perhaps the term "polynomial functor" is misleading here. It is easy to show that the category of Σ-algebras, $\mathsf{Alg}(\Sigma)$, is isomorphic to the category of $\mathbb{P}$-algebras, $\mathbf{Set}^{\mathbb{P}}$. For each Σ-algebra $\mathcal{S}$ and each $f^{(n)} \in \Sigma$, we have the interpretation of $f^{(n)}$ in $\mathcal{S}$,

$$f^{(n)}_{\mathcal{S}} : S^n \longrightarrow S\,.$$

Hence, there is a unique $\mathbb{P}$-algebra structure map $\sigma : \mathbb{P}S \to S$ making the diagram below commute.

$$\begin{tikzcd}
S^n \arrow[r, tail] \arrow[dr, "f^{(n)}_{\mathcal{S}}"'] & \coprod_{f^{(n)}\in\Sigma} S^n \arrow[d, dotted, "\sigma"] \\
 & S
\end{tikzcd}$$

Conversely, any $\langle S, \sigma\rangle$ in $\mathbf{Set}^{\mathbb{P}}$ corresponds to a Σ-algebra with $f^{(n)}_{\mathcal{S}}$ given by

$$S^n \rightarrowtail \coprod_{f^{(n)}\in\Sigma} S^n \xrightarrow{\alpha} S.$$

It's easy to see that Σ-homomorphisms are $\mathbb{P}$-homomorphisms, and vice-versa, so that this correspondence is an isomorphism of categories

$$\mathsf{Alg}(\Sigma) \cong \mathbf{Set}^{\mathbb{P}}.$$

Besides providing motivation for the approach of this dissertation, this example should convince the reader that algebras for an endofunctor are familiar territory. Sets and operations on sets are familiar enough, and these structures gave rise to the notion of universal algebras. The categorical notion of algebras for an endofunctor is simply a generalization of universal algebras, as we've seen here.

EXAMPLE 1.1.6. Let Z be a set and consider the $\mathbf{Set}$ functor

$$\Gamma A = Z \times A + 1.$$

An algebra for this functor consists of a pair $\langle A, \alpha\rangle$ where $\alpha : Z \times A + 1 \to A$. We decompose α into two maps,

$$*_\alpha : Z \times A \longrightarrow A, \text{ and}$$
$$()_\alpha : 1 \longrightarrow A.$$

A homomorphism from the Γ-algebra $\langle A, \alpha\rangle$ to $\langle B, \beta\rangle$ is a set function

$$f : A \longrightarrow B$$

such that, for all $z \in Z$ and $a \in A$,

$$f(z *_\alpha a) = z *_\beta f(a),$$
$$f(()_\alpha) = ()_\beta.$$

We will see in Example 1.5.6 that the initial algebra for this functor is the collection of all finite streams over Z, which we denote $Z^{<\omega}$. We can see now that $Z^{<\omega}$ is a Γ-algebra, with the structure map given by

$$\begin{aligned}\mathsf{push}: Z \times Z^{<\omega} &\longrightarrow Z^{<\omega}, \text{ and}\\ ():1 &\longrightarrow Z^{<\omega},\end{aligned}$$

where push returns the result of pushing a new letter onto a stream and $()$ returns the empty stream. More specifically,

$$\mathsf{push}(x, \sigma : n \longrightarrow Z) = \lambda k\,.\begin{cases} x & \text{if } k = 0\\ \sigma(k-1) & \text{else}\end{cases}$$

and $()$ is the unique map $0 \longrightarrow Z$.

1.1.3. Some examples of coalgebras. The dual category of coalgebras for an endofunctor may seem less familiar. In this section, we will give a few common examples of $\mathbf{Set}_\Gamma$ for a variety of endofunctors on $\mathbf{Set}$. In many these examples, the reader should notice that the structure map $\alpha : A \to \Gamma A$ acts as a *destructor*. It takes an element of the coalgebra and decomposes the element into its constituent parts. This is a common feature of coalgebras and this point of view is dual to the point of view that algebras are objects together with combinatory principles. However, the examples of Kripke models (Example 1.1.10) and topological spaces (Example 1.1.12) show that one can take talk of destructors too seriously.

Example 1.1.7. Consider the set functor

$$\Gamma A = Z \times A$$

for a fixed set Z. A coalgebra for this functor consists of a set A and a structure map

$$\alpha : A \longrightarrow Z \times A.$$

Equivalently, a coalgebra is given by a set A and two maps

$$\begin{aligned}h_\alpha : A &\longrightarrow Z, \text{ and}\\ t_\alpha : A &\longrightarrow A.\end{aligned}$$

Given any such coalgebra, each $a \in A$ gives rise to an infinite stream over Z, namely the stream

$$h_\alpha(a),\ h_\alpha \circ t_\alpha(a),\ h_\alpha \circ t_\alpha^2(a), \ldots$$

So, for any Γ-coalgebra $\langle A, \alpha\rangle$, we can define a mapping ! from A to the collection of streams over Z, Z^ω, by defining

$$!(a) = \lambda n\,.\,h_\alpha \circ t_\alpha^n(a).$$

It is worth noting, however, that this map is *not* necessarily one-to-one. Distinct elements of A may give rise to the same stream. For instance, consider the coalgebra $\langle A,\, \alpha\rangle$ where

$$A = \{a, b, c\}$$

and

$$\begin{aligned}\alpha(a) &= \langle 17,\, c\rangle,\\ \alpha(b) &= \langle 17,\, c\rangle,\\ \alpha(c) &= \langle 17,\, c\rangle.\end{aligned}$$

Then, one can see from the above definition of !, that

$$!(a) =!(b) =!(c).$$

Indeed, each of the elements of A maps to the constant 17 map.

We will see in Example 1.5.19 that the function ! is defined by *corecursion* on the collection of streams Z^ω.

A homomorphism between two Γ-coalgebras, $\langle A,\, \langle h_\alpha,\, t_\alpha\rangle\rangle$ and $\langle B,\, \langle h_\beta,\, t_\beta\rangle\rangle$ is a map $f\colon A \rightarrow B$ satisfying

$$\begin{aligned}h_\alpha(a) &= h_\beta(f(a)),\\ f(t_\alpha(a)) &= t_\beta(f(a)).\end{aligned}$$

The map ! is an example of such a homomorphism.

Example 1.1.8. Consider again the functor

$$\Gamma A = Z \times A + 1$$

from Example 1.1.6. A coalgebra for this functor consists of a set A together with a map

$$\alpha\colon A \longrightarrow Z \times A + 1\,.$$

So, each element a of such a coalgebra $\langle A,\, \alpha\rangle$ either maps to $*$, the unique element of 1, or to an ordered pair $\langle z,\, a'\rangle$, where $z \in Z$ and $a' \in A$. We can again interpret the coalgebras as collections of streams over Z if we allow each stream to be finite or infinite (above, we mapped coalgebras to collections of infinite streams). If $\alpha(a) = *$, then we take a to represent the empty stream. Otherwise, $\alpha(a) = \langle z,\, a'\rangle$ for some z and a'. Let σ' be the stream represented by a'. We say that a represents the stream $\mathsf{push}(z, a')$, where push is the stream with head z and tail a'. In this way, we define a mapping

$$!\colon A \longrightarrow Z^{\leq\omega}$$

satisfying

$$!(a) = \begin{cases} () & \text{if } \alpha(a) = * \\ \mathsf{push}(z, !(a')) & \text{else} \end{cases}$$

This map is again defined corecursively and is described in detail in Example 1.5.21. We mention it here to give the reader an intuition for the Γ-coalgebras. A Γ-coalgebra is a collection of finite and infinite streams over Z.

A homomorphism between two Γ-coalgebras must satisfy the same equations as in Example 1.1.7, if $\alpha(a) \in Z \times A$, and, if $\alpha(a) = *$, then $\beta(f(a)) = *$.

EXAMPLE 1.1.9. Let $\mathbb{P}$ be a polynomial functor on **Set**, which we'll write as

$$\mathbb{P}(A) = \coprod_{i<\omega} Z_i \times A^i.$$

A $\mathbb{P}$ coalgebra consists of a set A, together with a structure map $\alpha : A \to \mathbb{P}(A)$. Given such a coalgebra $\langle A, \alpha\rangle$, for each $a \in A$, define $\mathsf{br}(a)$ to be the unique i such that

$$\alpha(a) \in Z_i \times A^i.$$

We call the elements of $\pi_{A^i} \circ \alpha(a)$ the *children of* a. We denote the jth child,

$$\pi_j \circ \pi_{A^i} \circ \alpha(a),$$

by $\mathsf{child}_j(a)$. We call $\pi_{Z_i} \circ \alpha(a)$ the *label of* a, denoted $\mathsf{label}(a)$. In this way, we think of a Γ-coalgebra as a collection of labeled trees. Each element $a \in A$ is the root of a tree, where the immediate subtrees have the children of a as roots. The number of children is given by $\mathsf{br}(a)$, and the set of valid labels of a is given by $Z_{\mathsf{br}(a)}$. Take this description of coalgebras as trees as purely motivational for now — there will be more discussion on this in Example 1.5.22.

Examples 1.1.7 and 1.1.8 give a detailed account of two polynomial functors. In the former example, each node of the "tree" is labeled with an element of Z and has exactly one child. In the latter, each node has either 0 or 1 child. If it has 0 children, it is unlabeled (or, if you prefer, labeled with $*$). If it has 1 child, it is labeled with an element of Z, as before.

EXAMPLE 1.1.10. Given a set of atomic propositions **AtProp**, we can define an infinitary modal language $\mathcal{L}(\mathbf{AtProp})$ to be the least class containing **AtProp** and closed under the rules

- $\top \in \mathcal{L}(\mathbf{AtProp})$.
- If $\phi \in \mathcal{L}(\mathbf{AtProp})$, then so is $\neg\phi$ and $\Diamond\phi$.
- If $S \subset \mathcal{L}(\mathbf{AtProp})$, then $\bigwedge S \in \mathcal{L}(\mathbf{AtProp})$.

A Kripke model for the language $\mathcal{L}(\mathbf{AtProp})$ is given by a pair $\mathfrak{A} = \langle A, \alpha\rangle$, where A is a set and

$$\alpha : A \longrightarrow \mathcal{P}(A) \times \mathcal{P}(\mathbf{AtProp}).$$

The idea is that the first component of $\alpha(s)$ is the set of worlds accessible to s and the second component is the set of atomic propositions that hold in s. Accordingly, one defines a satisfaction relation $\models_{\mathfrak{A}}$ by the following:

- $a \models_{\mathfrak{A}} \top$.
- $a \models_{\mathfrak{A}} \phi$ for $\phi \in \mathbf{AtProp}$ iff $\phi \in \pi_2 \circ \alpha(a)$.
- $a \models_{\mathfrak{A}} \neg\phi$ iff $a \not\models_{\mathfrak{A}} \phi$.
- $a \models_{\mathfrak{A}} \Diamond\phi$ iff there is some $b \in \pi_1 \circ \alpha(a)$ such that $b \models_{\mathfrak{A}} \phi$.
- $a \models_{\mathfrak{A}} \bigwedge S$ iff $a \models_{\mathfrak{A}} \phi$ for each $\phi \in S$.

So, we see that Kripke models can be viewed as coalgebras for a particular functor in a straightforward manner, and that the resulting satisfaction relation comes directly from the coalgebraic structure map.

This example is covered in detail in [**BM96**, Chapter 11]. In the case that **AtProp** is empty, so the functor is just $A \mapsto \mathcal{P}(A)$, the coalgebras are called *Kripke structures* or *Kripke frames*. These are discussed in detail in [**Jac00, Che80, HC68**].

EXAMPLE 1.1.11. Fix a set of "inputs", $\mathcal{I}$ and let $\Gamma : \mathbf{Set} \to \mathbf{Set}$ be defined by

$$\Gamma S = (\mathcal{P}_{\mathsf{fin}} S)^{\mathcal{I}},$$

where $\mathcal{P}_{\mathsf{fin}}$ is the covariant finite powerset functor. A Γ-coalgebra $\langle S, \sigma \rangle$ can be regarded as a non-deterministic automaton over $\mathcal{I}$, where the structure map gives the transition function. Explicitly, for each state $s \in S$ and each input $i \in \mathcal{I}$, we write

$$s \xrightarrow{i} s'$$

just in case $s' \in \sigma(s)(i)$.

EXAMPLE 1.1.12. We take this example from [**Gum01b**].

Let A be a set. A *filter* on $\mathcal{P}A$ is a collection $\mathcal{U} \subseteq \mathcal{P}A$ if $\mathcal{U}$ is closed under finite intersections and supersets. In other words, $\mathcal{U}$ is a filter on $\mathcal{P}A$ just in case

- If $S, T \in \mathcal{U}$, then $S \cap T \in \mathcal{U}$, and
- If $S \in \mathcal{U}$ and $S \subseteq T$, then $T \in \mathcal{U}$.

We define a functor $\mathcal{F} : \mathbf{Set} \to \mathbf{Set}$ taking each set A to the collection of filters on A. If $f : A \to B$ is a map in **Set**, then for each $S \in \mathcal{P}A$, $\mathcal{F}f(S)$ is the filter generated by $\mathcal{P}f(S)$. See [**Gum01b**] for details on the functor $\mathcal{F}$.

Each topological space $\langle A, \mathcal{O}_A \rangle$ gives rise to an $\mathcal{F}$-coalgebra, as follows. We define the structure map $\alpha : A \to \mathcal{F}A$ on elements $a \in A$ by

$$\alpha(a) = \{S \subseteq A \mid \exists U \in \mathcal{O}_A \,.\, a \in U \subseteq S\}.$$

In other words, $\alpha(a)$ is the *neighborhood filter*[1] of a. It is easy to see that, if $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ are $\mathcal{F}$-coalgebras arising from topological spaces $\langle A, \mathcal{O}_A\rangle$ and $\langle B, \mathcal{O}_B\rangle$, respectively, then a map $f : A \to B$ is a coalgebra homomorphism just in case f is an open, continuous map. Thus, we have an inclusion

$$\mathbf{Top}^{\mathsf{open}} \longrightarrow \mathbf{Set}_{\mathcal{F}},$$

where $\mathbf{Top}^{\mathsf{open}}$ is the category of topological spaces and open, continuous maps.

EXAMPLE 1.1.13. Consider the functor $\Gamma A = Z \times A$ on the category **Top**, where Z is a fixed T_1 space (so points are topologically distinguishable). A Γ-coalgebra consists of a pair $\langle A, \alpha\rangle$ where A is a topological space and $\alpha : A \to \Gamma A$ is continuous. We will consider some carrier spaces for Γ-coalgebras and describe the Γ-structure map that can be imposed on the space.

Let I be the unit interval $[0, 1]$. Then a Γ-coalgebra with carrier I is just a path in the space $Z \times I$.

Let $\mathbf{2}$ denote the Sierpinski space and let $\sigma : \mathbf{2} \to \mathbb{R} \times \mathbf{2}$ be continuous. Let $\pi_1 \circ \sigma(0) = z_0$ and $\pi_2 \circ \sigma(0) = z_1$. For every open U containing z_0, $\pi_1^{-1}(U \times \mathbf{2})$ is open and so $z_1 \in U$. Hence, $z_0 = z_1$. So, a Γ-coalgebra with carrier $\mathbf{2}$ is specified by an element of Z and a map $\mathbf{2} \to \mathbf{2}$.

1.2. Structural features of $\mathcal{E}^\Gamma$ and $\mathcal{E}_\Gamma$

The categories $\mathcal{E}^\Gamma$ and $\mathcal{E}_\Gamma$ inherit much of the structure from the underlying category $\mathcal{E}$. In particular, $\mathcal{E}^\Gamma$ has whatever limits $\mathcal{E}$ has, and $\mathcal{E}_\Gamma$ has whatever colimits $\mathcal{E}$ has. If the functor Γ preserves colimits, then these are available in $\mathcal{E}^\Gamma$, and the dual result holds for $\mathcal{E}_\Gamma$. All of this is well-known and can be found in, for instance, [**Bor94**, Volume 2, Chapter 4], where these results are presented for algebras for a monad. The same proofs imply the following results for algebras for an endofunctor[2]. We present the main theorems here, without proof.

1.2.1. Creating (co)limits in categories of (co)algebras. The following definitions can be found in most standard category theory texts, including [**Lan71**].

DEFINITION 1.2.1. Let $G : \mathcal{C} \to \mathcal{C}'$ be a functor. We say that G *preserves $\mathcal{D}$-limits* if, for every diagram $J : \mathcal{D} \to \mathcal{C}$, whenever

$$\tau : A \Longrightarrow J$$

is a limiting cone, then

$$G\tau : GA \Longrightarrow G \circ J$$

[1] A *neighborhood* of a is any set $S \subseteq A$ containing an open set which contains a. We do not require that S itself is open.

[2] The key step is showing the existence of a structure map for the (co)limit. This step is essentially the same for both algebras for an endofunctor and algebras for a monad.

is a limiting cone for $G \circ J$.

We say that G *reflects $\mathcal{D}$-limits* if, for every $J : \mathcal{D} \to \mathcal{C}$, whenever

$$G\tau : GA \Longrightarrow G \circ J$$

is a limiting cone for $G \circ J$, then

$$\tau : A \Longrightarrow J$$

is a limiting cone for J.

Similarly, we define the statements G *preserves/reflects $\mathcal{D}$-colimits.*

If a functor preserves/reflects all (co)limits (regardless of the diagram category), we say the *functor preserves/reflects (co)limits.*

DEFINITION 1.2.2. We say that $G : \mathcal{C} \to \mathcal{C}'$ *creates $\mathcal{D}$-limits* if, whenever

$$J : \mathcal{D} \longrightarrow \mathcal{C}$$

and

$$\tau' : A' \Longrightarrow G \circ J$$

is a limiting cone in $\mathcal{C}'$, then there is a unique limiting cone

$$\tau : A \Longrightarrow J$$

in $\mathcal{C}$ such that $GA = A'$ and $G\tau = \tau'$.

Similarly, we define the statements G *creates $\mathcal{D}$-colimits* and G *creates (co)limits.*

So, if a functor $G : \mathcal{C} \to \mathcal{C}'$ creates $\mathcal{D}$-limits, then $\mathcal{C}$ has "as many" $\mathcal{D}$-limits as $\mathcal{C}'$ does. It is easy to see that if G creates $\mathcal{D}$-limits, then G reflects $\mathcal{D}$-limits. Also, if G creates $\mathcal{D}$-limits and $\mathcal{C}'$ has all $\mathcal{D}$-limits (is $\mathcal{D}$-complete), then G also preserves $\mathcal{D}$-limits and $\mathcal{C}$ is $\mathcal{D}$-complete.

DEFINITION 1.2.3. Additionally, we say that G *preserves regular epis* if, whenever p is a regular epi, then $G(p)$ is a regular epi.

Similarly, we define G *reflects regular epis.*

More generally, we define G *preserves/reflects maps of type* Θ, where Θ is some class of arrows (say, regular monos, isomorphisms, etc.)

It is worth noting that preservation of regular epis is weaker than preservation of coequalizers. If G preserves coequalizers, then any coequalizer diagram

$$B \underset{g}{\overset{f}{\rightrightarrows}} A \xrightarrow{q} Q$$

is taken to a coequalizer diagram

$$GB \underset{Gg}{\overset{Gf}{\rightrightarrows}} GA \xrightarrow{Gq} GQ \ .$$

If G preserves regular epis, however, we can only conclude that Gq is a coequalizer for some pair of maps. We cannot conclude that Gq is the coequalizer of Gf and Gg.

THEOREM 1.2.4. *Let $\mathcal{E}$ and $\Gamma : \mathcal{E} \to \mathcal{E}$ be given. The algebraic forgetful functor*

$$U : \mathcal{E}^{\Gamma} \longrightarrow \mathcal{E}$$

creates limits. Dually, the coalgebraic forgetful functor

$$U : \mathcal{E}_{\Gamma} \longrightarrow \mathcal{E}$$

creates colimits.

We interpret this theorem as saying that $\mathcal{E}^{\Gamma}$ has whatever limits $\mathcal{E}$ has, and that, furthermore, these limits are computed in $\mathcal{E}$. We apply this result in Section 1.5, for instance, to conclude that the initial coalgebra (final algebra, resp.) are trivial if $\mathcal{E}$ has an initial object (final object, resp.).

EXAMPLE 1.2.5. Let $\mathcal{E}$ have all κ-indexed products and let $\{\langle A_i, \alpha_i \rangle\}_{i \in \kappa}$ be an κ-indexed collection of Γ-algebras. Then the product

$$\prod_{i \in \kappa} \langle A_i, \alpha_i \rangle$$

is defined in $\mathcal{E}^{\Gamma}$ and is given by

$$\langle \prod_{i \in \kappa} A_i, \langle \alpha_i \rangle_{i \in \kappa} \rangle,$$

where

$$\langle \alpha_i \rangle_{i \in \kappa} : \Gamma \textstyle\prod_{i \in \kappa} A_i \longrightarrow \textstyle\prod_{i \in \kappa} A_i$$

is the unique map such that, for all $i \in \kappa$,

$$\pi_i \circ \langle \alpha_i \rangle_{i \in \kappa} = \alpha_i.$$

This is a generalization of the statement that products of universal algebras are the products of the underlying sets, with operations determined pointwise.

EXAMPLE 1.2.6. Dually, let $\mathcal{E}$ have all κ-indexed coproducts and let $\{\langle A_i, \alpha_i \rangle\}_{i \in \kappa}$ be an κ-indexed family of Γ-coalgebras. We have a family of maps

$$A_i \xrightarrow{\alpha_i} \Gamma A_i \xrightarrow{\Gamma \kappa_i} \Gamma \textstyle\coprod_{i \in \kappa} A_i \ ,$$

inducing a structure map

$$\textstyle\coprod_{i \in \kappa} A_i \longrightarrow \Gamma \textstyle\coprod_{i \in \kappa} A_i \ .$$

It is easy to confirm that this coalgebra is a coproduct in $\mathcal{E}_{\Gamma}$.

1.2.2. Colimits in $\mathcal{E}^\Gamma$, limits in $\mathcal{E}_\Gamma$. Again, this theorem can be found in [**Bor94**, Volume 2, Chapter 4], where the result is proved for categories of algebras for a monad.

THEOREM 1.2.7. *Let $\mathcal{D}$ be a category and $\Gamma : \mathcal{E} \to \mathcal{E}$. If Γ preserves $\mathcal{D}$-colimits then the forgetful functor $U : \mathcal{E}^\Gamma \to \mathcal{E}$ creates such colimits. Similarly, the coalgebraic forgetful functor $U : \mathcal{E}_\Gamma \to \mathcal{E}$ creates any limits preserved by Γ.*

So, for instance, if Γ preserves coequalizers, then $\mathcal{E}^\Gamma$ has all coequalizers and these are created by U. Unfortunately, the preservation of coequalizers seems a strong condition. However, we will get considerable mileage out of a weaker condition: preservation of regular epis.

In the coalgebraic setting, one often wants that the forgetful functor preserves pullbacks along regular monos. Other authors have ensured that this condition holds by assuming that Γ preserves weak pullbacks, We take the shorter path to the goal and assume that Γ preserves the appropriate pullbacks, since other weak pullbacks do not play a central role in this thesis. Applying Theorem 1.2.7, we have the following useful corollary.

COROLLARY 1.2.8. *If Γ preserves pullbacks along (regular) monos, then U creates pullbacks along (regular) monos.*

1.2.3. Factorizations of (co)algebras. In this section, we show how a category of (co)algebras can inherit a factorization system from its base category (see Appendix for a brief discussion of factorization systems). Explicitly, if $\mathcal{E}$ has regular epi-mono factorizations and kernel pairs and if Γ preserves regular epis, then the category of algebras $\mathcal{E}^\Gamma$ also has regular epi-mono factorizations, created by U. Furthermore, the forgetful functor preserves and reflects regular epis, monos and exact coequalizer sequences. Since every functor $\Gamma : \mathbf{Set} \to \mathbf{Set}$ preserves regular epis, this implies in particular that $\mathbf{Set}^\Gamma$ has regular epi-mono factorizations.

Dually,we learn that if $\mathcal{E}$ has epi-regular mono factorizations and cokernel pairs, and Γ preserves regular monos, then $\mathcal{E}_\Gamma$ has epi-regular mono factorizations, created by U.

The following lemma and its dual are useful in verifying that certain maps in $\mathcal{E}$ are homomorphisms.

LEMMA 1.2.9. *Suppose that $p : \langle A, \alpha\rangle \to \langle B, \beta\rangle$ be a Γ-algebra homomorphism and let $f : B \to C$ be given, where $C = U\langle C, \gamma\rangle$. Suppose further that Γp is epi. If $f \circ p$ is a homomorphism, then so is f.*

In particular, if Γ preserves epis (takes regular epis to epis, resp.) and p is an epi (regular epi, resp.) in $\mathcal{E}$, then f is a homomorphism whenever $f \circ p$ is.

PROOF. Consider Figure 1. A simple diagram chase confirms that

$$\gamma \circ \Gamma f \circ \Gamma p = f \circ \beta \circ \Gamma p.$$

Since Γp is an epi, f is a homomorphism. $\square$

$$\begin{tikzcd}
\Gamma A \arrow[r, two heads, "\Gamma p"] \arrow[d, "\alpha"'] & \Gamma B \arrow[r, "\Gamma f"] \arrow[d, "\beta"] & \Gamma C \arrow[d, "\gamma"] \\
A \arrow[r, "p"'] & B \arrow[r, "f"'] & C
\end{tikzcd}$$

FIGURE 1. If $f \circ p$ is a homomorphism, then so is f.

COROLLARY 1.2.10. *Let $i: \langle B, \beta\rangle \to \langle C, \gamma\rangle$ be a coalgebra homomorphism, and let $f: A \to B$ be a map in $\mathcal{E}$, where $A = U\langle A, \alpha\rangle$. If Γi is monic and $i \circ f$ a coalgebra homomorphism, then f is a coalgebra homomorphism.*

In particular, if Γ preserves monos (takes regular monos to monos, resp.) and i is mono (regular mono, resp.) in $\mathcal{E}$, then f is a homomorphism whenever $i \circ f$ is.

PROOF. By duality. $\square$

If Γ preserves epis, then $U: \mathcal{E}^\Gamma \to \mathcal{E}$ reflects strong epis, as can easily be verified. Lemma 1.2.11 gives the analogous claim for regular epis, which we will use to prove that $\mathcal{E}^\Gamma$ has regular epi-mono factorizations given certain conditions on $\mathcal{E}$ and Γ (see Theorem 1.2.13).

Throughout, we will prefer regular epi-mono factorization systems over strong epi-mono factorization systems, but this is largely a matter of choice. As one can see in explicitly in [**Kur00, Kur99**], the basic theorems go through just as easily with strong epis in the place of regular epis. We stick with the regular epis because of the connection between coequalizers and sets of equations in Chapter 3. For the sake of duality, we also stress regular monos in the coalgebraic cases.

LEMMA 1.2.11. *Let $\mathcal{E}$ have kernel pairs and $\Gamma: \mathcal{E} \to \mathcal{E}$ take regular epis to epis. Then*

$$U: \mathcal{E}^\Gamma \longrightarrow \mathcal{E}$$

reflects regular epis.

PROOF. Let $p: \langle A, \alpha\rangle \to \langle B, \beta\rangle$ be a map in $\mathcal{E}^\Gamma$ and suppose that p is a regular epi in $\mathcal{E}$. Let

$$\langle K, \kappa\rangle \overset{k_1}{\underset{k_2}{\rightrightarrows}} \langle A, \alpha\rangle$$

be the kernel pair of p and suppose $f:\langle A,\, \alpha\rangle \twoheadrightarrow \langle C,\, \gamma\rangle$ coequalizes the kernel pair (see Figure 2). Since U preserves kernel pairs, p is the coequalizer of k_1 and k_2 in $\mathcal{E}$. Hence,

$$
\begin{tikzcd}
 & & \Gamma B \arrow[dr, bend left, "\Gamma g"] \arrow[dd] & \\
\Gamma K \arrow[r, shift left, "\Gamma k_1"] \arrow[r, shift right, "\Gamma k_2"'] \arrow[dd, "\kappa"'] & \Gamma A \arrow[ur, bend left, twoheadrightarrow, "\Gamma p"] \arrow[rr] \arrow[dd, "\alpha"'] & & \Gamma C \arrow[dd, "\gamma"] \\
 & & B \arrow[dr, bend left, "g"] & \\
K \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] & A \arrow[ur, bend left, twoheadrightarrow, "p"] \arrow[rr, "f"'] & & C
\end{tikzcd}
$$

FIGURE 2. U reflects regular epis.

there is a unique map $g:B\to C$ in $\mathcal{E}$ such that $g\circ p = f$. Apply Lemma 1.2.9. $\square$

The next theorem (about factorizations in $\mathcal{E}^\Gamma$) proves especially useful, as we will see. Thus, it is worthwhile to attach a name to the conditions that we assume on $\mathcal{E}$. That these conditions are part of the definition of regular category suggests the following definition.

DEFINITION 1.2.12. A category $\mathcal{C}$ is *almost regular* if $\mathcal{C}$ has kernel pairs and regular epi-mono factorizations (we don't require that kernel pairs have coequalizers or that regular epis are stable under pullbacks).

Dually, a category with cokernel pairs and epi-regular mono factorizations is *almost co-regular.*

THEOREM 1.2.13. *Let $\mathcal{E}$ have be almost regular and let $\Gamma:\mathcal{E}\to\mathcal{E}$ preserve regular epis. Then $\mathcal{E}^\Gamma$ has regular epi-mono factorizations, preserved and reflected by $U:\mathcal{E}^\Gamma\to\mathcal{E}$.*

PROOF. Let $f:\langle A,\, \alpha\rangle \to \langle B,\, \beta\rangle$ and take the regular epi-mono factorization, $f = i\circ p$, in $\mathcal{E}$ (as in Figure 3). Because Γp is regular and hence strong, there is a structure map γ, as shown making both i and p homomorphisms. Since the forgetful functor reflects regular epis and monos, we see that $i \circ p$ is a regular epi-mono factorization in $\mathcal{E}^\Gamma$, obviously preserved by U.

Since regular epi-mono factorizations are unique up to isomorphism, this is sufficient to conclude that U preserves all regular epi-mono factorizations. $\square$

The following definition is found in [**Bor94**, Volume 2, Chapter 2], where exact sequences in regular categories are described in detail.

$$\begin{tikzcd} \Gamma A \arrow[r, "\Gamma p"] \arrow[d, "\alpha"'] & \Gamma C \arrow[r, "\Gamma i"] \arrow[d, dotted, "\gamma"'] & \Gamma B \arrow[d, "\beta"] \\ A \arrow[r, "p"'] & C \arrow[r, tail, "i"'] & B \end{tikzcd}$$

FIGURE 3. Regular epi-mono factorization in $\mathcal{E}^\Gamma$.

DEFINITION 1.2.14. A diagram of the form

$$K \underset{e_2}{\overset{e_1}{\rightrightarrows}} A \xrightarrow{q} Q$$

is an *exact sequence* if q is the coequalizer of e_1 and e_2, and e_1, e_2 is the kernel pair of q.

We also call a diagram of the form

$$E \xrightarrow{i} A \underset{c_2}{\overset{c_1}{\rightrightarrows}} D$$

an *exact sequence* if i is the equalizer of c_1 and c_2 and c_1, c_2 the cokernel pair of i.

COROLLARY 1.2.15. *Let $\mathcal{E}$ be almost regular and let $\Gamma : \mathcal{E} \to \mathcal{E}$ preserve regular epis. Then $U : \mathcal{E}^\Gamma \to \mathcal{E}$ preserves and reflects regular epis, monos and exact sequences.*

PROOF. By Theorem 1.2.13 and uniqueness of regular epi-mono factorizations, U preserves regular epis and monos.

Because U preserves and reflects kernel pairs and regular epis, and regular epis are coequalizers of their kernel pairs, U preserves and reflects exact sequences. ☐

REMARK 1.2.16. It is important to note that all of these theorems dualize for categories of coalgebras in an obvious way. Explicitly, if $\mathcal{E}$ is almost co-regular and Γ preserves regular monos, then $\mathcal{E}_\Gamma$ inherits epi-regular mono factorizations from $\mathcal{E}$.

1.3. Subalgebras

We have a notion of subobject for any category: namely, a subobject of A is an equivalence class of monics with codomain A (see Appendix). This definition applies to the categories $\mathcal{E}^\Gamma$ and $\mathcal{E}_\Gamma$ to yield:

> A subobject of a Γ-(co)algebra $\langle A, \alpha \rangle$ is an equivalence class of monic homomorphisms
>
> $$\langle B, \beta \rangle \rightarrowtail \langle A, \alpha \rangle \ .$$

In categories of algebras, we are most interested in those subobjects of $\langle A, \alpha\rangle$ which are preserved by U. These can be understood as subobjects of A which are closed under the algebraic operations.

We postpone the discussion of subcoalgebras until Section 2.2. There, we take the position that subcoalgebras are best understood as the dual of quotients of algebras. Consequently, we are interested in *regular* subobjects of a coalgebra.

DEFINITION 1.3.1. Let $\langle A, \alpha\rangle$ be a Γ-algebra. A *subalgebra* of $\langle A, \alpha\rangle$ is a subobject

$$i : \langle B, \beta\rangle \rightarrowtail \langle A, \alpha\rangle$$

such that $Ui : B \rightarrowtail A$ is a subobject of A (Ui is a mono in $\mathcal{E}$).

For each Γ-algebra, there are three related posets. First, there is the poset $\mathsf{Sub}_{\mathcal{E}^\Gamma}(\langle A, \alpha\rangle)$. This consists of equivalence classes of monos

$$\langle B, \beta\rangle \overset{i}{\rightarrowtail} \langle A, \alpha\rangle$$

in $\mathcal{E}^\Gamma$. We also have the poset $\mathsf{Sub}_{\mathcal{E}}(A)$ of subobjects of the carrier of $\langle A, \alpha\rangle$. Lastly, we have the poset $\mathsf{SubAlg}(\langle A, \alpha\rangle)$ of subalgebras of $\langle A, \alpha\rangle$. This poset has, as objects, equivalence classes of monos

$$\langle B, \beta\rangle \overset{i}{\rightarrowtail} \langle A, \alpha\rangle$$

such that Ui is mono in $\mathcal{E}$. Evidently,

$$\mathsf{SubAlg}(\langle A, \alpha\rangle) \subseteq \mathsf{Sub}_{\mathcal{E}^\Gamma}(\langle A, \alpha\rangle).$$

In the categories in which we are most interested, this inclusion is an isomorphism.

THEOREM 1.3.2. *If $\mathcal{E}$ is almost regular and Γ preserves regular epis, then*

$$\mathsf{SubAlg}(\langle A, \alpha\rangle) \cong \mathsf{Sub}_{\mathcal{E}^\Gamma}(\langle A, \alpha\rangle).$$

PROOF. If Γ preserves regular epis, then U preserves monos (Corollary 1.2.15). □

We note that any **Set** functor Γ preserves regular epis and so

$$\mathsf{SubAlg}(\langle A, \alpha\rangle) \cong \mathsf{Sub}_{\mathcal{E}^\Gamma}(\langle A, \alpha\rangle).$$

We turn our attention to the relationship between $\mathsf{SubAlg}(\langle A, \alpha\rangle)$ and $\mathsf{Sub}_{\mathcal{E}}(A)$ (hereafter, denoted $\mathsf{Sub}(A)$). In order to determine the structure of the category $\mathsf{SubAlg}(\langle A, \alpha\rangle)$, we look at the structure of $\mathsf{Sub}(A)$. We will show that $\mathsf{SubAlg}(\langle A, \alpha\rangle)$ inherits much of the structure of $\mathsf{Sub}(A)$. In order to make this clear, we define a functor

$$U_\alpha : \mathsf{SubAlg}(\langle A, \alpha\rangle) \longrightarrow \mathsf{Sub}(A).$$

This functor takes a subalgebra $\langle B, \beta\rangle$ to its carrier B as a subobject of A.

REMARK 1.3.3. The functor U_α is a component of a natural transformation between contravariant functors

$$U : \mathsf{SubAlg} \Longrightarrow \mathsf{Sub},$$

but we will not make use of this fact.

THEOREM 1.3.4. *The functor U_α is an injection. In particular, for any $B \overset{i}{\rightarrowtail} A = U\langle A, \alpha\rangle$, there is at most one structure map $\beta : \Gamma B \to B$ making i a homomorphism.*

PROOF. Let $\langle B, \beta\rangle$ and $\langle C, \gamma\rangle$ be subalgebras of $\langle A, \alpha\rangle$ and suppose

$$U_\alpha(\langle B, \beta\rangle) = U_\alpha(\langle C, \gamma\rangle).$$

Then B and C are equal as subobjects of A. Without loss of generality, assume $B = C$ and let the inclusion be given by $i : B \rightarrowtail A$. By assumption, i is a homomorphism, so

$$i \circ \beta = \alpha \circ \Gamma i = i \circ \gamma,$$

so $\beta = \gamma$. □

THEOREM 1.3.5. *U_α creates meets. Thus, if $\mathsf{Sub}(A)$ is a complete lattice, then so is the category $\mathsf{SubAlg}(\langle A, \alpha\rangle)$.*

PROOF. This follows from the fact that $U : \mathcal{E}^\Gamma \to \mathcal{E}$ creates limits (Theorem 1.2.4). □

1.3.1. Subalgebras generated by a subset. Let $\langle A, \alpha\rangle$ be a Γ-algebra and P a subobject of A (in $\mathcal{E}$). In this section, we discuss the least subalgebra containing P, which we denote $\langle P\rangle_\alpha$ or just $\langle P\rangle$. As we will see, this subalgebra exists under fairly weak assumptions. We give two constructions of $\langle P\rangle$. The first construction (Theorem 1.3.6) requires that $\mathsf{Sub}_{\mathcal{E}}(A)$ is a complete lattice. The second construction requires that $\mathcal{E}$ is almost regular and Γ preserves regular epis. Further, we assume that the algebraic forgetful functor $U : \mathcal{E}^\Gamma \to \mathcal{E}$ is monadic (equivalently, U has a left adjoint). See Section 2.1.2 for a discussion of the left adjoint of U.

We understand the functor $\langle -\rangle_\alpha$ in terms of adjointness. Specifically, if each subobject P of A is contained in a least subalgebra $\langle P\rangle_\alpha$ of $\langle A, \alpha\rangle$, then we have an adjoint pair $\langle -\rangle_\alpha \dashv U_\alpha$ (dropping the subscript when convenient). We call the subalgebra $\langle P\rangle_\alpha$ the *subalgebra generated by P*.

THEOREM 1.3.6. *Let $\langle A, \alpha\rangle$ be a Γ-algebra and suppose that $\mathsf{Sub}(A)$ is a complete lattice (say, if $\mathcal{E}$ is complete and well-powered). Then the functor*

$$U_\alpha : \mathsf{SubAlg}(\langle A, \alpha\rangle) \longrightarrow \mathsf{Sub}(A)$$

has a left adjoint

$$\langle -\rangle_\alpha : \mathsf{Sub}(A) \longrightarrow \mathsf{SubAlg}(\langle A, \alpha\rangle).$$

Proof. We will explicitly construct $\langle - \rangle$. Let $i\colon P \rightarrowtail A$ be a subobject of A. We take the intersection of all the subalgebras containing P,

$$\langle P \rangle = \bigwedge_{P \subseteq Q} \langle Q,\, \rho \rangle.$$

$\square$

The following theorem is an alternate construction of $\langle P \rangle$ that applies in the categories in which we are most interested. We also include it because the resulting construction is very natural: $\langle P \rangle$ arises as the factorization of

$$FP \longrightarrow \langle A,\, \alpha \rangle,$$

where $F \dashv U$. See Section 2.1 for a discussion of such adjoint functors.

Theorem 1.3.7. *Suppose $\mathcal{E}$ is almost regular, Γ preserves regular epis and that U has a left adjoint F (i.e., Γ is a varietor, in the sense of* [**AP01**]. *Let $\langle A,\, \alpha \rangle$ be a Γ-algebra and P be a subobject of A. Then we have an adjoint pair*

$$\begin{tikzcd} \mathsf{Sub}(A) \arrow[r, bend left, "\langle - \rangle_\alpha"] \arrow[r, phantom, "\perp"] & \mathsf{SubAlg}(\langle A,\, \alpha \rangle) \arrow[l, bend left, "U_\alpha"] \end{tikzcd} .$$

Proof. Let ε be the counit of the adjunction $F \dashv U$. Let $i\colon P \rightarrowtail A$ be the inclusion of P into A and take the regular epi-mono factorization $j \circ p$ of $\varepsilon_\alpha \circ Fi$, shown in Figure 4.

$$\begin{tikzcd} & FQ \arrow[dr, bend left, "Fk"] \arrow[dd] & \\ FP \arrow[ur, bend left, "Fl"] \arrow[rr] \arrow[dd, "p"'] & & FA \arrow[dd, "\varepsilon_\alpha"] \\ & \langle Q,\, \nu \rangle \arrow[dr, "k"] & \\ \langle P \rangle \arrow[ur, dotted] \arrow[rr, tail, "j"'] & & \langle A,\, \alpha \rangle \end{tikzcd}$$

Figure 4. The construction of $\langle P \rangle$ as a regular epi-mono factorization.

We first show that $P \leq U\langle P \rangle$. It suffices to show that $j \circ p \circ \eta_P = i$ (see Figure 5). One calculates

$$\begin{aligned} j \circ p \circ \eta_P &= U\varepsilon_\alpha \circ UFi \circ \eta_P \\ &= U\varepsilon_\alpha \circ \eta_A \circ i = i. \end{aligned}$$

The inequality $P \leq U\langle P\rangle$ is the unit of the adjunction, of course.

$$\begin{tikzcd}
& UFP \arrow[dr, "p", two heads] & \\
P \arrow[ur, "\eta_P"] \arrow[dr, "i"', tail] & & U_\alpha\langle P\rangle_\alpha \arrow[dl, "j", tail] \\
& A &
\end{tikzcd}$$

FIGURE 5. P is contained in $U\langle P\rangle$.

Let $k\colon\langle Q,\, \nu\rangle \rightarrowtail \langle A,\, \alpha\rangle$ be a subalgebra of $\langle A,\, \alpha\rangle$ and $P \leq Q$ (with inclusion l). We wish to show that $\langle P\rangle \leq \langle Q,\, \nu\rangle$. We have

$$\begin{aligned} k \circ \varepsilon_\nu \circ Fl &= \varepsilon_\alpha \circ Fk \circ Fl \\ &= \varepsilon_\alpha \circ Fj = j \circ p, \end{aligned}$$

and so, since p is strong, we have the factorization desired. □

As we will see, in the dual category $\mathcal{E}_\Gamma$, given a coalgebra $\langle A,\, \alpha\rangle$ and a subobject $P \leq A$, the natural construction yields the greatest subcoalgebra contained in P. In other words, we have a right adjoint to the analogous forgetful functor

$$U_\alpha\colon \mathsf{SubCoalg}(\langle A,\, \alpha\rangle) \longrightarrow \mathsf{Sub}(A).$$

We discuss this adjoint pair in Section 2.2.

The adjoint pair $\langle -\rangle_\alpha \dashv U_\alpha$ gives rise to a closure operator

$$U_\alpha\langle -\rangle_\alpha\colon \mathsf{Sub}(A) \longrightarrow \mathsf{Sub}(A)$$

on the subobjects of A. This operator takes a subobject P and closes it under the operations (structure map) of the algebra. The unit of the monad is the inclusion

$$P \leq U_\alpha\langle P\rangle_\alpha.$$

The multiplication is the identity

$$U_\alpha\langle U_\alpha\langle P\rangle_\alpha\rangle_\alpha = U_\alpha\langle P\rangle_\alpha.$$

As Theorem 1.3.5 showed, if $\mathsf{Sub}(A)$ is complete, then so is $\mathsf{SubAlg}(\langle A,\, \alpha\rangle)$. General results in order theory tell one how to define joins on $\mathsf{SubAlg}(\langle A,\, \alpha\rangle)$, but it is worth stating the result explicitly: Given a collection

$$\{\langle B_i,\, \beta_i\rangle\}_{i\in I}$$

of subalgebras of $\langle A,\, \alpha\rangle$, their join is given by

$$\bigvee\langle B_i,\, \beta_i\rangle = \langle\bigvee B_i\rangle_\alpha.$$

1.4. Congruences

We generalize the notions introduced above to binary relations here. It should be clear that these notions generalize to n-ary relations, but we do not do so explicitly. Binary relations deserve special attention since they arise as the kernels of homomorphisms.

Recall that a relation on $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ is a triple $\langle\langle R, \rho\rangle, r_1, r_2\rangle$ where

$$r_1 : \langle R, \rho\rangle \longrightarrow \langle A, \alpha\rangle,$$
$$r_2 : \langle R, \rho\rangle \longrightarrow \langle B, \beta\rangle$$

are jointly monic (see the Appendix for a brief review of relations). This definition works whether we are speaking of algebras or coalgebras, of course. Again, we will want to pay particular attention to those relations of $\mathcal{E}^\Gamma$ which are preserved by U. We postpone the discussion of relations in $\mathcal{E}_\Gamma$ until Section 2.5, where we introduce bisimulations.

Definition 1.4.1. Let $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ be Γ-algebras. A relation

$$\langle\langle R, \rho\rangle, r_1, r_2\rangle$$

on $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ is a *pre-congruence* if $\langle R, r_1, r_2\rangle$ is a relation on A and B.

Let $\mathsf{PreCong}(\langle A, \alpha\rangle, \langle B, \beta\rangle)$ be the poset of pre-congruences on $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$. We will often abbreviate this category as $\mathsf{PreCong}(\alpha, \beta)$. Again, we relate this category to the related posets of relations, $\mathsf{Rel}_{\mathcal{E}^\Gamma}(\alpha, \beta)$ and $\mathsf{Rel}_{\mathcal{E}}(A, B)$.

We also often abbreviate the product of two (co)algebras,

$$\langle A, \alpha\rangle \times \langle B, \beta\rangle,$$

as $\alpha \times \beta$.

Theorem 1.4.2. *If $\mathcal{E}$ is almost regular, has binary products and Γ preserves regular epis, then*

$$\mathsf{PreCong}(\alpha, \beta) \cong \mathsf{Rel}_{\mathcal{E}^\Gamma}(\alpha, \beta) = \mathsf{Sub}_{\mathcal{E}}(\alpha \times \beta).$$

Proof. A relation $\langle R, \rho\rangle$ on $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ is a subalgebra of the algebra $\langle A, \alpha\rangle \times \langle B, \beta\rangle$. Because U preserves both products and monos, we see that R is a subobject of $A \times B$ and hence a relation in $\mathcal{E}$. Thus, R is a pre-congruence. $\square$

In Section 1.3, we defined a forgetful functor taking subalgebras of $\langle A, \alpha\rangle$ to their carrier as a subobject of A. We analogously define a forgetful functor here

$$U_{\alpha,\beta} : \mathsf{PreCong}(\alpha, \beta) \longrightarrow \mathsf{Rel}(A, B),$$

taking a pre-congruence $\langle R, \rho\rangle$ to its carrier R as a relation on A and B.

In fact, $U_{\alpha,\beta}$ is just

$$U_{\alpha\times\beta}\colon \mathsf{SubAlg}(\alpha\times\beta) \longrightarrow \mathsf{Sub}_{\mathcal{E}}(A\times B).$$

Thus, from Theorems 1.3.4 and 1.3.5, we have the following corollaries.

COROLLARY 1.4.3. *The functor $U_{\alpha,\beta}$ is an inclusion of $\mathsf{PreCong}_{\mathcal{E}^\Gamma}(\alpha,\beta)$ into $\mathsf{Rel}_{\mathcal{E}}(A,B)$. In other words, the structure map on an algebraic relation is unique.*

COROLLARY 1.4.4. *The functor $U_{\alpha,\beta}$ creates meets. Hence, if $\mathsf{Rel}(A,B)$ (that is, $\mathsf{Sub}(A\times B)$) is complete, then so is $\mathsf{PreCong}(\alpha,\beta)$ ($=\mathsf{SubAlg}(\alpha\times\beta)$).*

REMARK 1.4.5. Again, we have a natural transformation (natural in both components) between the contravariant bifunctors

$$U\colon \mathsf{PreCong} \Longrightarrow \mathsf{Rel}.$$

The functor

$$\langle -\rangle_{\alpha\times\beta}\colon \mathsf{Sub}(A\times B) \longrightarrow \mathsf{SubAlg}(\alpha\times\beta),$$

if it exists, gives a construction of least pre-congruences. That is, given any relation R on A and B (any subobject of $A\times B$), $\langle R\rangle_{\alpha\times\beta}$ is the least pre-congruence on $\langle A,\,\alpha\rangle$ and $\langle B,\,\beta\rangle$ containing R (i.e., the least subalgebra of $\alpha\times\beta$ containing R). When we view $\langle -\rangle_{\alpha\times\beta}$ as a functor

$$\mathsf{Rel}(A,B) \longrightarrow \mathsf{PreCong}(\alpha,\beta),$$

we will sometimes write $\langle -\rangle_{\alpha,\beta}$. We drop the subscripts entirely if the meaning of $\langle -\rangle$ is clear from context.

We are often interested in pre-congruences on an algebra $\langle A,\,\alpha\rangle$ by itself — that is, in the category $\mathsf{PreCong}(\alpha,\alpha)$. These pre-congruences can be viewed as sets of equations (see Remark 1.4.7), which will play a central role in Chapter 3. The following principle is useful for reasoning about $\langle R\rangle_{\alpha,\alpha}$.

THEOREM 1.4.6. *Let $\mathcal{E}$ be finitely complete, and $\Gamma\colon\mathcal{E}\to\mathcal{E}$ be given. Let $\langle A,\,\alpha\rangle$ be a Γ-algebra and R a relation on A. Let $f\colon\langle A,\,\alpha\rangle\to\langle B,\,\beta\rangle$ be a Γ-homomorphism. Then the following diagram (in $\mathcal{E}$) commutes*

$$R \rightrightarrows A \xrightarrow{f} B \tag{1}$$

iff the diagram (in $\mathcal{E}^\Gamma$) below commutes.

$$\langle R\rangle \rightrightarrows \langle A,\,\alpha\rangle \xrightarrow{f} \langle B,\,\beta\rangle \tag{2}$$

PROOF. If (2) commutes, then the fact that R is contained in $U\langle R\rangle$ ensures that (1) commutes.

Suppose, conversely, that (1) commutes and take the kernel pair $\langle K,\,\kappa\rangle$ of f in $\mathcal{E}^\Gamma$. Because the forgetful functor $U\colon\mathcal{E}^\Gamma\to\mathcal{E}$ creates kernel pairs, K is the kernel pair of f in $\mathcal{E}$, so R is a subrelation of K. Since $\langle R\rangle$ is the least pre-congruence containing R, $\langle R\rangle$ is contained in $\langle K, \kappa\rangle$. Thus, f coequalizes $\langle R\rangle \rightrightarrows \langle A, \alpha\rangle$. □

REMARK 1.4.7. Let $\langle A, \alpha\rangle$ and R be given as in the statement of Theorem 1.4.6. We can view R as a set of equations on A — namely, R corresponds to the set of equations

$$\{r_1(x) = r_2(x) \mid x \in R\}.$$

We say that B satisfies the equations in R under the assignment f if f equalizes r_1 and r_2. That is,

$$B, f \models_A R$$

just in case the diagram

$$R \rightrightarrows A \xrightarrow{f} B$$

commutes.

In these terms, we can restate Theorem 1.4.6 as follows: For any homomorphism $f : \langle A, \alpha\rangle \to \langle B, \beta\rangle$,

$$B, f \models_A R \text{ iff } \langle B, \beta\rangle, f \models_{\langle A, \alpha\rangle} \langle R\rangle.$$

See Chapter 3 for a proper development of equations for categories $\mathcal{E}^\Gamma$.

1.4.1. Exact categories of algebras. Throughout this section, we assume that $\mathcal{E}$ is finitely complete and has regular epi-mono factorizations, so that $\mathcal{E}$ is, in particular, "almost regular". We also assume that $\Gamma : \mathcal{E} \to \mathcal{E}$ preserves regular epis, so that $\mathcal{E}^\Gamma$ inherits regular epi-mono factorization from $\mathcal{E}$ (Theorem 1.2.13).

A congruence is a pre-congruence which is an equivalence relation. Because pre-congruences are relations in two different categories (both $\mathcal{E}^\Gamma$ and $\mathcal{E}$), there is apparent ambiguity in this definition. We will show that the ambiguity is illusory — a pre-congruence which is an equivalence relation in $\mathcal{E}$ is also an equivalence relation in $\mathcal{E}^\Gamma$, and vice versa.

Because $U : \mathcal{E}^\Gamma \to \mathcal{E}$ creates limits and regular epi-mono factorizations, one has the following theorem.

THEOREM 1.4.8. *The forgetful functor $U_{\alpha,\beta}$ preserves the following structure of* $\mathsf{PreCong}(\alpha, \beta)$.

(1) *For any composable pre-congruences $\langle R, \rho\rangle \in \mathsf{PreCong}(\alpha, \beta)$ and $\langle S, \sigma\rangle \in \mathsf{PreCong}(\beta, \gamma)$,*

$$U_{\alpha,\gamma}(\langle S, \sigma\rangle \circ \langle R, \rho\rangle) = S \circ R.$$

(2) *For any pre-congruence $\langle R, \rho\rangle$ on $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$,*

$$U_{\beta,\alpha}(\langle R, \rho\rangle^0) = R^0$$

(where R^0 is the twist of relation R — see the Appendix).

(3) For any algebra $\langle A, \alpha\rangle$,

$$U_{\alpha,\alpha}\Delta_{\langle A,\,\alpha\rangle} = \Delta_A$$

(where Δ_A *is equality on* A *— see the Appendix).*

PROOF. 2 and 3 are obvious. For the first, we use the fact that U creates pullbacks and finite regular epi-mono source factorizations. It creates the latter because it creates regular epi-mono factorizations and products (and because $\mathcal{E}$ has finite products). □

DEFINITION 1.4.9. A pre-congruence on $\langle A, \alpha\rangle$ which is also an equivalence relation is a *congruence.*

The following corollary shows that it is enough for $\langle R, \rho\rangle$ to be a pre-congruence such that R is an equivalence relation (in $\mathcal{E}$).

COROLLARY 1.4.10. *Let* $\langle A, \alpha\rangle$ *be a* Γ*-algebra and let* $\langle R, \rho\rangle$ *be a pre-congruence. Then* $\langle R, \rho\rangle$ *is a congruence iff* R *is an equivalence relation in* $\mathcal{E}$*.*

PROOF. By Theorem 1.4.8 and the fact that $U_{\alpha,\alpha}$ is full. □

The remainder of the section is intended to give an example of reasoning about algebras in a categorical setting. We present a generalization of a standard theorem in the study of universal algebras. It states that one can take coequalizers of congruences in $\mathcal{E}^\Gamma$ (i.e., that $\mathcal{E}^\Gamma$ is exact — see Definition A.4.4). We will prove that this theorem holds in a variety of categories and for a variety of functors — namely, it holds in any exact category if the endofunctor Γ preserves exact sequences. The standard theorem about algebras over **Set** is an easy corollary.

THEOREM 1.4.11. *Let* $\mathcal{E}$ *be an exact category with binary products and* $\Gamma\colon\mathcal{E}\to\mathcal{E}$ *preserve exact sequences (coequalizers of kernel pairs). The category* $\mathcal{E}^\Gamma$ *is also exact.*

PROOF. Let p be a regular epi in $\mathcal{E}$. Take the kernel pair of p,

$$\bullet \rightrightarrows \bullet \xrightarrow{p} \bullet.$$

Since Γ preserves exact sequences, we see that Γp is again a regular epi, so Γ preserves all regular epis. Hence, U preserves and reflects monos, regular epis and exact sequences (Theorem 1.2.15 — note that any regular category has regular epi-mono factorizations [**Bor94**, Proposition 2.2.1]).

Let

$$\langle R, \rho\rangle \rightrightarrows \langle A, \alpha\rangle$$

be an equivalence relation in $\mathcal{E}^\Gamma$. Since

$$\mathsf{PreCong}(\langle A, \alpha\rangle) \cong \mathsf{Rel}_{\mathcal{E}^\Gamma}(\langle A, \alpha\rangle),$$

$\langle R, \rho\rangle$ is a congruence, and so R is an equivalence relation in $\mathcal{E}$. Since $\mathcal{E}$ is exact and R is an equivalence relation, R is the kernel pair of a regular epi q, as shown below.

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] & A \arrow[r, "q"] & Q \end{tikzcd}$$

This diagram is an exact sequence (in an exact category, an equivalence relation is always the kernel pair of its coequalizer), so its image under Γ is again an exact sequence.

Hence, the top row of the diagram below is a coequalizer.

$$\begin{tikzcd} \Gamma R \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\rho"'] & \Gamma A \arrow[r, "\Gamma q"] \arrow[d, "\alpha"'] & \Gamma Q \arrow[d, dotted, "\nu"] \\ R \arrow[r, shift left] \arrow[r, shift right] & A \arrow[r, "q"'] & Q \end{tikzcd}$$

A simple diagram chase shows that there is a unique ν making the right hand square commute. Because U reflects regular epis, q is a regular epi in $\mathcal{E}^\Gamma$. □

THEOREM 1.4.12. *Let $\mathcal{E}$ be a exact category with binary products and suppose $\mathcal{E}$ satisfies the weak axiom of choice. The category $\mathcal{E}^\Gamma$ is also exact.*

PROOF. It is easy to show that every exact sequence is an absolute coequalizer (see the proof of [**Bor94**, Volume 2, Theorem 4.3.5], for instance), and so is preserved by every functor. □

1.4.2. Least congruence constructions. Given an algebra $\langle A, \alpha\rangle$ and a relation R on A, one is often interested in the least congruence $\overline{R}$ containing R. These is the least relation on A such that the quotient $A/\overline{R}$ can be taken in $\mathcal{E}^\Gamma$. In this section, we will show that, if $\mathcal{E}$ is exact with binary products and Γ preserves exact sequences, then we can define a functor

$$\mathsf{Rel}(A, A) \longrightarrow \mathsf{Cong}(\alpha)$$

(where $\mathsf{Cong}(\alpha)$ is the category of congruences on $\langle A, \alpha\rangle$) taking a relation to its least congruence. This material is included just to complete our development of congruences. It is a well-known result.

THEOREM 1.4.13. *Let $\mathcal{E}$ be exact, with binary products, and Γ preserve exact sequences (and, hence, regular epis). Then the inclusion functor*

$$U_{\alpha,\alpha} : \mathsf{Cong}(\alpha) \longrightarrow \mathsf{Rel}(A, A)$$

has a left adjoint.

PROOF. We know from Theorem 1.4.11 that $\mathcal{E}^\Gamma$ is exact. We construct a functor $K : \mathsf{PreCong}(\alpha, \alpha) \to \mathsf{Cong}(\alpha)$, left adjoint to the evident inclusion functor. This construction works in any exact category, just by taking a relation to the kernel pair of its coequalizer. Now, given a relation R on A and a congruence $\langle S, \sigma\rangle$ on $\langle A, \alpha\rangle$, we see that

$$R \leq S \Leftrightarrow \langle R\rangle \leq \langle S, \sigma\rangle \Leftrightarrow K\langle R\rangle \leq \langle S, \sigma\rangle.$$

□

1.5. Initial algebras and final coalgebras

In categories of algebras and coalgebras, the presence of initial objects and terminal objects, respectively, plays an important role. Initial algebras satisfy the induction proof principle and definition by recursion, while final coalgebras enjoy the analogous principles of coinduction and definition by corecursion. In this section, we discuss these principles and the nature of initial algebras and final coalgebras as least and greatest fixed points, respectively, for the endofunctor Γ.

Recall that in a category $\mathcal{C}$, an *initial object* A is an object such that, for any $Y \in \mathcal{C}$, there is exactly one arrow $A \longrightarrow Y$. Dually, a *final* or *terminal object* Z has the property that each $Y \in \mathcal{C}$ has exactly one arrow $Y \longrightarrow Z$. Any two initial (final) objects are clearly isomorphic. If $\mathcal{C}$ is a poset, then an initial object is just $\perp$ and a final object is just $\top$.

For algebras, the initial algebra is an important object, coming equipped with certain "proof principles". However, the final algebra is typically dull. If $\mathcal{E}$ has a final object, 1, then, for any functor Γ, there is a final Γ-algebra, namely $\langle 1, !_1\rangle$, where $!_1$ is the unique map $\Gamma 1 \longrightarrow 1$. This is a corollary to the fact that U creates limits (Theorem 1.2.4). For **Set**, for example, this means that the one point algebra is always the final algebra. Dually, if $\mathcal{E}$ has an initial object, 0, then $\langle 0, !_0\rangle$ is the initial coalgebra, where $!_0 : 0 \to \Gamma 0$. In **Set**, then, the empty coalgebra is always the initial coalgebra (whatever the endofunctor $\Gamma : \mathbf{Set} \to \mathbf{Set}$).

1.5.1. Fixed points for a functor. Given a functor $\Gamma : \mathcal{E} \to \mathcal{E}$, we can consider the collection of fixed points of Γ, i.e., those $C \in \mathcal{E}$ such that $\Gamma C \cong C$. Such objects can be regarded as both Γ-algebras and Γ-coalgebras. Let $\mathsf{Fix}(\Gamma)$ be the full subcategory of $\mathcal{E}^\Gamma$ consisting of those algebras for which the structure map is an isomorphism. Equivalently, we could take the same full subcategory of $\mathcal{E}_\Gamma$, since Γ algebra homomorphisms between fixed points are Γ coalgebra homomorphisms and vice-versa. Lambek's lemma [**Lam70**] states, first, that the initial algebra (final coalgebra), if it exists, is in $\mathsf{Fix}(\Gamma)$. It easily follows that the initial algebra is also

$$\begin{tikzcd} \Gamma A \arrow[r, "\Gamma !"] \arrow[d, "\alpha"'] & \Gamma^2 A \arrow[r, "\Gamma\alpha"] \arrow[d, "\Gamma\alpha"'] & \Gamma A \arrow[d, "\alpha"] \\ A \arrow[r, "!"'] & \Gamma A \arrow[r, "\alpha"'] & A \end{tikzcd}$$

FIGURE 6. Initial algebras are fixed points.

initial in $\mathsf{Fix}(\Gamma)$, and the final coalgebra is final in $\mathsf{Fix}(\Gamma)$ (See Section 1.5.4 for a discussion of the unique homomorphism between the two).

LEMMA 1.5.1 (Lambek's lemma). *If $\langle A, \alpha \rangle$ is an initial Γ-algebra, then α is an isomorphism. Dually, the structure map of a final coalgebra is also an isomorphism.*

PROOF. Because $\langle A, \alpha \rangle$ is initial, there is a unique homomorphism ! from $\langle A, \alpha \rangle$ to the algebra

$$\langle \Gamma A, \, \Gamma\alpha : \Gamma^2 A \longrightarrow \Gamma A \rangle.$$

In Figure 6, the bottom composite is the identity, by the uniqueness condition for initiality. Because ! is a Γ-homomorphism, the left hand square commutes. Consequently,

$$! \circ \alpha = \Gamma\alpha \circ \Gamma ! = \mathsf{id}_{\Gamma A} .$$

□

This result brings out a central fact about initial algebras/final coalgebras — namely, they are the same thing as initial fixed points/final fixed points for an endofunctor. In many cases (though, not all cases), they are in fact least fixed points/greatest fixed points for the endofunctor in the usual sense. In this respect, at least, initial algebras should seem familiar objects of study. Languages specified by a syntax are given as a least fixed point for an endofunctor on **Set**, for instance. In particular, the modal language $\mathcal{L}(\mathbf{AtProp})$ was described earlier as a least fixed point. Hence, we may regard this and similar languages as initial algebras for suitable functors.

Lambek's lemma also gives us a negative result regarding initial algebras and final coalgebras. If a functor has no fixed points, then it has no initial algebra or final coalgebra. Of course, the power set functor, $\mathcal{P}:\mathbf{Set}\to\mathbf{Set}$, has no fixed points (due to Cantor's theorem). Consequently, there is no initial algebra/final coalgebra for this functor *as a functor on* **Set**.

However, there is a closely related functor for which the initial algebra and final coalgebra both exist and are well known. Consider the category **SET** of all sets and classes (without the axiom of foundation). We can extend the functor $\mathcal{P}$ to a functor (also denoted $\mathcal{P}$) on this category taking each class to its class of subsets (note: subsets, not subclasses). See [**BM96**] for details on the extension of set-based functors to the category **SET**. The initial algebra for this functor is the class **WF** of well-founded sets, with identity as the structure map. The final coalgebra for this functor is **NWF**, the category of sets with the anti-foundation axiom, again with identity as the structure map. For additional reading on fixed points for $\mathcal{P}$, see [**BM96**], [**Acz88**] and [**Tur96**].

For existence theorems for both initial algebras and final coalgebras, see [**Bar92**]. James Worrell extends this discussion in [**Wor00**].

1.5.2. Induction and recursion. See also [**JR97**] for a nice exposition of this material.

The principle of definition by recursion is an explicit application of the property of initiality. Given any Γ-algebra $\langle B, \beta\rangle$, there is a unique homomorphism from the initial Γ-algebra $\langle I, \iota\rangle$ to $\langle B, \beta\rangle$ (just by definition of initiality). This categorical property leads to familiar principles in application.

Example 1.5.2. For instance, consider the successor functor $S\colon\mathbf{Set}\to\mathbf{Set}$ taking a set X to the set $X + 1$ (the disjoint union of X and $\{*\}$). The initial algebra for this functor is $\langle\mathbb{N}, [s, 0]\rangle$, where

$$s(n) = n + 1,$$
$$0(*) = 0.$$

Indeed, the initial algebra for S in any category with $+$ is called the *natural numbers object (NNO).*

To justify this terminology, consider the usual statement of definition by recursion on $\mathbb{N}$. Namely, given any set A together with an element $a \in A$ and a map $f\colon A\to A$, there is a unique map $!\colon\mathbb{N}\to A$ such that

$$!(0) = a,$$
$$!(n + 1) = f(!(n)).$$

(We'll ignore the apparently stronger statement of definition by recursion with parameters for now.) But, specifying a and f is just the same as specifying a map

$$[f, a]\colon A + 1 \longrightarrow A.$$

Also, the equations above exactly require the diagram below

$$\begin{tikzcd}
\mathbb{N}+1 \arrow[r, "!+\mathrm{id}"] \arrow[d, "{[s,0]}"'] & A+1 \arrow[d, "{[f,a]}"] \\
\mathbb{N} \arrow[r, "!"'] & A
\end{tikzcd}$$

to commute, i.e., require that ! is an S-homomorphism.

EXAMPLE 1.5.3. In Example 1.5.2, we showed that the statement that $\mathbb{N}$ is an initial algebra for the successor functor is equivalent to the statement that for each $a \in A$ and $f{:}A \to A$, there is a unique map $!{:}\mathbb{N} \to A$ such that

$$!(0) = a,$$
$$!(n+1) = f(!(n)).$$

Of course, one usually wants to define more complicated functions recursively. In this example, we will show that the statement that $\mathbb{N}$ is an initial algebra for S allows the recursive definition of functions with parameters. Specifically, given two functions,

$$g{:}A \longrightarrow A, \text{ and}$$
$$h{:}A \times A \longrightarrow A$$

we will show that there exists a unique $f{:}\mathbb{N} \times A \to A$ such that

$$f(0, a) = g(a), \tag{3}$$
$$f(s(n), a) = h(f(n, a), a). \tag{4}$$

Initiality guarantees maps with domain $\mathbb{N}$, so we will define a map $\widetilde{f}{:}\mathbb{N} \to A^A$ and show that its transpose is the map f we desire. To define such a $\widetilde{f}$ by recursion, we must find a structure map $\alpha{:}A^A + 1 \to A^A$ such that the unique homomorphism $\mathbb{N} \longrightarrow A^A$, guaranteed by initiality, is the $\widetilde{f}$ we desire.

Let α be defined by

$$\alpha(*) = g{:}A \longrightarrow A,$$
$$\alpha(k) = \lambda a \,.\, h(k(a), a) \qquad \text{for all } k{:}A \longrightarrow A.$$

Then, by initiality, there is a unique $\widetilde{f}$ such that

$$\widetilde{f}(0) = g,$$
$$\widetilde{f}(n+1) = \lambda a \,.\, h(\widetilde{f}(n)(a), a).$$

Consequently, the transpose of $\widetilde{f}$ satisfies (3) and (4).

In a similar manner, we can show that there is a unique $f{:}\mathbb{N} \to A$ such that

$$f(0, a) = g(a),$$
$$f(s(n), a) = h(f(n, a), n, a).$$

For this, we must define a structure map α for $(A \times \mathbb{N})^A$ so that the unique map $\widetilde{f}$ making the square below commute

$$\begin{tikzcd}
\mathbb{N}+1 \arrow[r, "\widetilde{f}+\mathsf{id}_1"] \arrow[d, "{[s,0]}"'] & (A \times \mathbb{N})^A + 1 \arrow[d, "\alpha"] \\
\mathbb{N} \arrow[r, "\alpha"'] & (A \times \mathbb{N})^A
\end{tikzcd}$$

satisfies the appropriate equations. This is left as an exercise for the reader.

Example 1.5.4. In the category **Poset**, the natural numbers object (initial algebra for S) is again the algebra $\langle \mathbb{N}, [s,0] \rangle$. As a poset, we take the trivial ordering: $x \leq y$ iff $x = y$.

The natural numbers with the standard ordering (which we denote ω) is also an initial algebra in **Poset**, but for a different functor. Consider the lifting functor $-_\perp$: **Poset**$\to$**Poset** that takes a poset and adjoins a new bottom element. The initial algebra for this functor is ω. The structure map

$$\omega_\perp \longrightarrow \omega$$

takes $\perp$ to 0 and takes each $n \in \omega$ to $s(n)$.

Example 1.5.5. Example 1.5.2 shows that $\mathbb{N}$ is an initial algebra for the polynomial functor S. Here, we examine the general case.

Let $\mathbb{P}$ be a polynomial functor and define a signature Σ so that $\mathbb{P}$ is the polynomial functor for Σ, i.e., so that

$$\mathbb{P}(A) = \coprod_{f^{(n)} \in \Sigma} A^n.$$

Let $\mathcal{L}(\Sigma)$ be the collection of all Σ-terms. Explicitly, $\mathcal{L}(\Sigma)$ is the least set such that the following holds:

- If $f^{(n)} \in \Sigma$ and $\tau_1, \ldots, \tau_n \in \mathcal{L}(\Sigma)$, then $f^{(n)}(\tau_1, \ldots, \tau_n) \in \mathcal{L}(\Sigma)$.

Of course, this entails in particular that any constants (that is, zero-ary function symbols) of Σ are in $\mathcal{L}(\Sigma)$. One should also notice that, if Σ has no constants, then $\mathcal{L}(\Sigma)$ is empty.

We impose an algebraic structure on $\mathcal{L}(\Sigma)$ in the obvious manner. For each $f^{(n)} \in \Sigma$, we must define a map

$$\mathcal{L}(\Sigma)^n \longrightarrow \mathcal{L}(\Sigma)\ .$$

Let $\tau_1, \ldots, \tau_n$ be in $\mathcal{L}(\Sigma)$ and define the interpretation of $f^{(n)}$ to be

$$\langle \tau_1, \ldots, \tau_n \rangle \mapsto f(\tau_1, \ldots, \tau_n).$$

It is routine to check that $\mathcal{L}(\Sigma)$ together with this structure map is an initial $\mathbb{P}$-algebra.

There is another description of the initial $\mathbb{P}$-algebra. Namely, we consider $\mathcal{L}(\Sigma)$ as a family of finitely branching, Σ-labeled trees, subject to the condition:

- If a node is labeled $f^{(n)}$, then the node has exactly n children (consequently, a node labeled with a constant $c^{(0)}$ is a leaf).

We have, then, that $\mathcal{L}(\Sigma)$ is the least collection of trees such that

- For each $f^{(n)} \in \Sigma$ and each $\tau_1, \ldots, \tau_n \in \mathcal{L}(\Sigma)$, the tree with root labeled $f^{(n)}$ and with children $\tau_1, \ldots, \tau_n$ is in $\mathcal{L}(\Sigma)$.

Again, we stress that, in particular, for each constant $c^{(0)}$ in Σ, the tree consisting of a node (with no children) labeled $c^{(0)}$ is in $\mathcal{L}(\Sigma)$.

EXAMPLE 1.5.6. We show now that $\langle Z^{<\omega}, [\mathsf{push}, ()]\rangle$ is an initial algebra for $\Gamma A = Z \times A + 1$ (see Example 1.1.6). Let $\langle A, \langle *_\alpha, ()_\alpha\rangle\rangle$ be any Γ-algebra. Define a sequence of maps $!^n : Z^n \to A$ as follows:

$$!^0(()) = ()_\alpha,$$
$$!^{n+1}(\mathsf{push}(z, \sigma)) = z *_\alpha !(\sigma).$$

We take $! : Z^{<\omega} \to A$ to be $\bigcup_{i=1}^{\omega} !^n$. It is easy to see that $!$ is a Γ-homomorphism and that it is unique.

The principle of definition by Γ-recursion can thus be stated: For any set A, element $a \in A$ and map $f : Z \times A \to A$, there is a unique $! : Z^\omega \to A$ such that

$$!(()) = a,$$
$$!(\mathsf{push}(z, \sigma)) = f(z, !(\sigma)).$$

We also have a least fixed point definition of $Z^{<\omega}$, arising from the discussion of Section 1.5.1. Namely, $Z^{<\omega}$ is the least collection such that

- $() \in Z^{<\omega}$;
- If $z \in Z$ and $\sigma \in Z^{<\omega}$, then $\mathsf{push}(z, \sigma) \in Z^{<\omega}$.

This description of $Z^{<\omega}$ agrees with the description of an initial algebra for a polynomial functor from Example 1.5.5 (allowing that the terms are interpreted as elements of $Z^{<\omega}$).

This concludes our discussion of recursion. We now turn to the related property of induction.

The principle of induction allows one to conclude that a particular property P holds of all of the elements of an initial algebra if P is closed under the operations of the algebra. We will show in this section how the principle of induction is a minimality condition which follows from initiality. We will include some explicit examples of how the minimality condition leads to a familiar induction principle.

LEMMA 1.5.7. *Let $\langle I, \iota\rangle$ be an initial Γ-algebra. Then any map into $\langle I, \iota\rangle$ is a regular epi.*

PROOF. Let $f\colon\langle A, \alpha\rangle \twoheadrightarrow \langle I, \iota\rangle$ be given and let $!\colon\langle I, \iota\rangle \twoheadrightarrow \langle A, \alpha\rangle$ be the homomorphism guaranteed by initiality. Then, by the uniqueness part of initiality, $f\circ !$ is the identity, so f is a regular epi. □

As one can see, Lemma 1.5.7 is not about initial algebras, per se, but rather is true of any initial object in any category. The next theorem is an abstract statement of the principle of induction. Again, it is a corollary to a general statement about initial objects.

THEOREM 1.5.8. *If $\langle I, \iota\rangle$ is an initial Γ-algebra, then $\langle I, \iota\rangle$ is minimal, i.e.,*

$$\mathsf{Sub}_{E^\Gamma}(\langle I, \iota\rangle) = \{\langle I, \iota\rangle\}.$$

So, in particular, $\langle I, \iota\rangle$ has no proper subalgebras (subobjects preserved by U).

PROOF. Let $\langle P, \rho\rangle$ be a subobject of $\langle I, \iota\rangle$, with homomorphic inclusion

$$i\colon\langle P, \rho\rangle \longrightarrow \langle I, \iota\rangle\,.$$

By Lemma 1.5.7, i is a regular epi and so is an isomorphism. □

Let $\langle A, \alpha\rangle$ be an algebra. We say that a subobject $P \overset{i}{\rightarrowtail} A$ of A is *closed under α* if there is a structure map

$$\rho\colon\Gamma P \longrightarrow P$$

such that

$$i\colon\langle P, \rho\rangle \longrightarrow \langle A, \alpha\rangle$$

is a homomorphism. In other words, P is closed under α just in case

$$P = U_\alpha\langle P\rangle_\alpha$$

(that is, P is closed under the closure operator $U_\alpha\langle -\rangle_\alpha$). The property of minimality ensures that any predicate closed under α exhausts the entire algebra. It is useful to see a couple of explicit examples.

We also say that a subobject P closed under α is an *inductive predicate.*

REMARK 1.5.9. The category of all subobjects of A closed under α is isomorphic to $\mathsf{SubAlg}(\langle A, \alpha\rangle)$, so we aren't really introducing a new concept here. Instead, we introduce new language that allows one to see that the principle of induction for initial algebras is the usual principle of induction for the familiar examples of initial algebras. When discussing induction, it is conventional to speak of predicates which are closed under certain operations, rather than to speak of subalgebras. We follow that convention, although there is no practical difference between the two.

EXAMPLE 1.5.10. As discussed in previously, $\langle \mathbb{N}, [s, 0] \rangle$ is an initial algebra for the successor functor S. A subset P of $\mathbb{N}$ is closed under $[s, 0]$ just in case there is a $\rho : P + 1 \to P$ making the diagram below commute.

$$\begin{tikzcd} P + 1 \arrow[r, "i+\mathsf{id}_1"] \arrow[d, "\rho"'] & \mathbb{N} + 1 \arrow[d, "{[s,0]}"] \\ P \arrow[r, "i"'] & \mathbb{N} \end{tikzcd}$$

This means that

$$\begin{aligned} i \circ \rho(*) &= 0, \\ i \circ \rho(n) &= s(n) \qquad \text{for each } n \in P. \end{aligned}$$

In other words, P is a subalgebra of $\mathbb{N}$ just in case $0 \in P$ and whenever $n \in P$, also $s(n) \in P$. From Theorem 1.5.8, we see that if P contains 0 and is closed under s, then $P = \mathbb{N}$. So, Theorem 1.5.8 yields induction on the natural numbers in the usual sense.

EXAMPLE 1.5.11. Consider again the initial algebra $\mathcal{L}(\Sigma)$ for a fixed signature Σ (see Example 1.5.5). One can confirm that minimality on $\mathcal{L}(\Sigma)$ entails the following proof principle: If, for each $f^{(n)} \in \Sigma$

$$\forall \tau_1, \ldots, \tau_n (\Phi(\tau_1) \wedge \ldots \wedge \Phi(\tau_n)) \to \Phi(f(\tau_1, \ldots, \tau_n)),$$

then $\Phi(\tau)$ for all $\tau \in \Sigma$. Note that, as usual, if a predicate Φ is closed under function application, then, in particular, Φ holds for every constant.

Call a tree *well-founded* if the relation "is a descendant of" is well-founded in the usual sense — that is, if there are no infinite paths in the tree. Then, one can show, using the above principle of induction, that every element of $\mathcal{L}(\Sigma)$ (viewed as trees — see Example 1.5.5) is well-founded. We omit this proof, since it requires a more explicit representation of trees than we give here.

EXAMPLE 1.5.12. Let $P \subseteq Z^{<\omega}$. Then, P is inductive just in case

$$\begin{aligned} () &\in P, \\ \mathsf{push}(z, \sigma) &\in P \qquad \text{if } z \in Z \text{ and } \sigma \text{ in P.} \end{aligned}$$

If P satisfies these conditions, then $P = Z^{<\omega}$.

EXAMPLE 1.5.13. As mentioned previously, the class **WF** of well-founded sets with identity is an initial algebra for the functor $\mathcal{P} : \mathbf{SET} \to \mathbf{SET}$. It is useful to see what the principal of induction yields for this algebra. A predicate on **WF** is a subclass of **WF**. A predicate Φ is inductive iff whenever $\Phi(S)$ for all $S \in T$, then

$\Phi(T)$. Thus, induction says, for each predicate Φ,

$$\forall T(\forall S \in T\, \Phi(S) \to \Phi(T)) \to \forall T\, \Phi(T),$$

where the quantifiers here range over **WF**. Equivalently, we have, for each Φ,

$$\exists T \Phi(T) \to \exists T\, (\Phi(T) \wedge \forall S \in T \neg \Phi(S)).$$

In other words, the principle of induction on **WF** as an initial algebra is the usual foundation axiom. Put another way, the foundation axiom is equivalent to the assumption that the class of all sets is an initial algebra for $\mathcal{P}$ (although here, we've only shown one implication — see [**Tur96**] for the other).

It is worth mentioning that the property of minimality isn't unique to initial algebras. On the contrary, any algebra which is a quotient of the initial algebra is also minimal, and so satisfies an inductive proof principle. Conceptually, if $\langle A, \alpha \rangle$ is a quotient of the initial algebra, then each element of A can be picked out by a term (not necessarily uniquely). So, if the atomic elements (the interpretations of constants) satisfy a predicate and if the predicate is closed under term formation, then all of A satisfies the predicate.

THEOREM 1.5.14. *Let $\mathcal{E}$ be almost regular and Γ preserve regular epis and suppose that $\mathcal{E}^\Gamma$ has an initial object $\langle I, \iota \rangle$. An algebra $\langle A, \alpha \rangle$ is minimal iff the map $! : \langle I, \iota \rangle \twoheadrightarrow \langle A, \alpha \rangle$ is a regular epi.*

PROOF. Suppose $\langle A, \alpha \rangle$ is minimal. Take the regular epi-mono factorization $! = i \circ p$. Then i is an isomorphism and so $!$ is a regular epi.

On the other hand, suppose that

$$! : \langle I, \iota \rangle \longrightarrow \langle A, \alpha \rangle$$

is a regular epi and

$$i : \langle P, \rho \rangle \rightarrowtail \langle A, \alpha \rangle$$

is a mono. Then $i \circ !_\rho = !_\alpha$, so i is a regular epi and hence an isomorphism. $\square$

COROLLARY 1.5.15. *Let $\mathcal{E}$, Γ be as in Theorem 1.5.14. If $\langle A, \alpha \rangle$ is minimal, then α is a regular epi.*

PROOF. Since $\alpha \circ \Gamma ! = ! \circ \iota$ and the right hand side is the composite of two regular epis (ι is an isomorphism), α is a regular epi. $\square$

REMARK 1.5.16. The converse of Corollary 1.5.15 does not generally hold. That is, if α is a regular epi, then $\langle A, \alpha \rangle$ need not be minimal. Let $\langle F, \phi \rangle$ be the final Γ-coalgebra. Then $\langle F, \phi^{-1} \rangle$ is a Γ-algebra in which the structure map is a regular epi. Typically, however, $\langle F, \phi^{-1} \rangle$ is not minimal. Indeed, it is common that the initial algebra is a proper subalgebra of $\langle F, \phi^{-1} \rangle$.

1.5.3. Corecursion and coinduction. Dually, the unique homomorphism into the final coalgebra is said to be *defined by corecursion*. Definition by corecursion resembles a kind of "baseless recursion". However, it is important to keep in mind that corecursion gives a map into, not out of, the final coalgebra.

EXAMPLE 1.5.17. In Example 1.5.2, we showed that $\mathbb{N}$ forms an initial algebra for S. We can also describe the final coalgebra for the successor functor S. Take $\mathbb{N}$ and adjoin a point ∞. Call this set $\overline{\mathbb{N}}$. Define a S-coalgebra structure $p{:}\overline{\mathbb{N}}\rightarrow\overline{\mathbb{N}}+1$ on $\overline{\mathbb{N}}$ by

$$p(x) = \begin{cases} * & \text{if } x = 0 \\ n & \text{if } x = n+1 \\ \infty & \text{if } x = \infty. \end{cases}$$

The intuition here is that p is the predecessor function, taking ∞ to itself, $n+1$ to n and 0 to the "error condition", $*$.

If $\langle A,\, \alpha\rangle$ is any S-coalgebra, then we define $!\colon A\rightarrow\overline{\mathbb{N}}$ by

$$!(a) = \begin{cases} \mu n \,.\, \alpha^n(a) = * & \text{if this is defined} \\ \infty & \text{else} \end{cases}$$

The proof that ! is a coalgebra homomorphism and that it is unique is left to the reader. It is worth noting that $\overline{\mathbb{N}}$ is *not* the greatest fixed point for S (under inclusion). Of course, S doesn't have a greatest fixed point, since any infinite set is a fixed point.

EXAMPLE 1.5.18. Just as ω is an initial algebra for the lifting functor

$$-_\perp{:}\mathbf{Poset}\longrightarrow\mathbf{Poset},$$

$\omega+1$ is the final coalgebra for $\perp$. Similarly, the set $\overline{\mathbb{N}}$ is the final coalgebra for S.

EXAMPLE 1.5.19. In Example 1.1.7, we claimed that coalgebras for the functor $\Gamma A = Z\times A$ could be regarded as collections of infinite streams over Z. Here, we will make precise what we meant by that claim. We first take the collection of Z-streams, Z^ω, and impose a coalgebraic structure on it. Specifically, we consider the coalgebra $\langle Z^\omega,\, \langle h,\, t\rangle\rangle$, where

$$\begin{aligned} h(\sigma\colon\omega\longrightarrow Z) &= \sigma(0), \\ t(\sigma\colon\omega\longrightarrow Z) &= \lambda n \,.\, \sigma(n+1). \end{aligned}$$

Let $\langle A,\, \langle h_\alpha,\, t_\alpha\rangle\rangle$ be a Γ-coalgebra. We define a mapping,

$$!\colon A\longrightarrow Z^\omega,$$

as in Example 1.1.7, by

$$a \mapsto \lambda n \,.\, h_\alpha \circ t_\alpha^n(a).$$

Then one can easily confirm that ! is a homomorphism. Furthermore, any map $f : A \to Z^\omega$ satisfying

$$h_\alpha(a) = h(f(a)),$$
$$f(t_\alpha(a)) = t(f(a)).$$

must agree with !.

The principle of definition by Γ-corecursion can be stated thus: Given any set A and any pair of maps,

$$j : A \longrightarrow Z,$$
$$k : A \longrightarrow A,$$

there is exactly one map $! : A \to Z^\omega$ such that, for all $a \in A$,

$$j(a) = h(!(a)),$$
$$!(k(a)) = t(!(a)).$$

Remark 1.5.20. We have not discussed the initial algebra for $A \mapsto Z \times A$. There is good reason for this: it is trivial. That is, the initial algebra for this functor is just the algebra

$$\langle 0,\ Z \times 0 \rightarrowtriangle 0 \rangle.$$

Example 1.5.21. In Example 1.5.19, we made precise the claim that each coalgebra for $A \mapsto Z \times A$ could be considered a collection of streams over Z, as mentioned in Example 1.1.7. In this example, we will clarify the claim of Example 1.1.8, that every coalgebra for the functor $\Gamma A = Z \times A + 1$ can be regarded as a collection of (finite and infinite) streams over Z. In particular, we will impose a coalgebraic structure on $Z^{\leq\omega}$ and prove that the resulting coalgebra is final.

We define $\zeta : Z^{\leq\omega} \to Z \times Z^{\leq\omega} + 1$ by

$$\zeta(\sigma) = \begin{cases} * & \text{if } \sigma \in Z^0 \text{ (i.e., } \sigma = () : \emptyset \to Z) \\ \langle \sigma(0),\ \lambda n \,.\, \sigma(n+1) \rangle & \text{else} \end{cases}$$

Notice that, if $\sigma \in Z^\omega$, then $\lambda n \,.\, \sigma(n+1) \in Z^\omega$, but if $\sigma \in Z^{n+1}$, then $\lambda n \,.\, \sigma(n+1) \in Z^n$. In other words, if $\sigma \in Z^\omega$, then the "tail" of σ is again in Z^ω, while if $\sigma \in Z^{n+1}$, then the tail of σ is in Z^n.

Let $\langle A, \alpha \rangle$ be any Γ-coalgebra. We define h_α and t_α as $\pi_1 \circ \alpha$ and $\pi_2 \circ \alpha$, when these are defined. We define a map $! : A \to Z^{\leq\omega}$ by

$$!(a) = \begin{cases} () & \text{if } \alpha(a) = * \\ \lambda n \,.\, h_\alpha \circ t_\alpha^n(a) & \text{else.} \end{cases}$$

Notice that, in the second case, the resulting function may be defined only for certain n. More precisely, $!(a)$ may as a function whose domain is an initial segment of ω, i.e., an element of Z^n for some n.

We omit the details of confirming that ! is a homomorphism and that it is unique.

EXAMPLE 1.5.22. Let $\mathbb{P}$ be a polynomial functor,

$$\mathbb{P}(A) = \coprod_{i<\omega} Z_i A^i$$

on the category **Set**. From Example 1.5.11, we saw that the initial algebra for $\mathbb{P}$ can be viewed as a collection of well-founded labeled trees. The final coalgebra can similarly be viewed as the collection of all labeled trees (well-founded or not) with the same branching behavior as the initial algebra. To make this description precise, one needs a model of this collection of trees. While such a model can be described as a collection of sets of finite sequences, representing paths through the tree, closed under appropriate conditions, the details of such a description are more technical than illuminating and will be skipped here.

Alternatively, one could use Aczel's non-well-founded set theory to describe the final coalgebra as the (necessarily unique) set T such that

$$T = \mathbb{P}(T),$$

and use the identity as the structure map, an approach made popular by [**BM96**].

The dual of the principle of induction for initial algebras is that final coalgebras are *coalgebra!simple*, i.e., that they have no proper quotients. This property is best expressed as a property about the relations on final coalgebras.

REMARK 1.5.23. In fact, final coalgebras satisfy a stronger condition. If $\langle A, \alpha\rangle$ is a final Γ homomorphism and

$$p : \langle A, \alpha\rangle \twoheadrightarrow \langle B, \beta\rangle$$

is any (not necessarily regular) epi, then p is an isomorphism. We find that the condition of simplicity suffices for most of our purposes, however, and use it instead.

Just as any quotient of a minimal algebra is again minimal, any subcoalgebra of a simple coalgebra is again simple. Furthermore, if there is a final Γ-coalgebra $\langle F, \phi\rangle$, then any simple coalgebra is a (regular) subobject of $\langle F, \phi\rangle$, as the following corollary shows. Hence, a coalgebra $\langle A, \alpha\rangle$satisfies the principle of coinduction iff $\langle A, \alpha\rangle$is simple iff $\langle A, \alpha\rangle$is an *open* object of $\mathcal{E}_\Gamma$, in the sense of [**LM92**, Chapter IV.6].

COROLLARY 1.5.24. *Let $\mathcal{E}$ be almost co-regular factorizations and Γ preserve regular monos and let $\langle F, \phi\rangle$ be the final Γ-coalgebra. A coalgebra $\langle A, \alpha\rangle$ is simple iff $! : \langle A, \alpha\rangle \rightarrowtail \langle F, \phi\rangle$ is a regular mono.*

PROOF. This is the dual of Theorem 1.5.14. ☐

Typically, we view coinduction as a proof principle that says, if two elements of a simple coalgebra are related by a coalgebraic relation, then they are equal. This next theorem is a step to that proof principle, which we return to in Section 2.6.

THEOREM 1.5.25. *Let $\mathcal{E}$ have all coequalizers. The following are equivalent:*

(1) $\langle A, \alpha\rangle$ is simple.

(2) For any coalgebra $\langle B, \beta\rangle$, there is at most one map

$$\langle B, \beta\rangle \longrightarrow \langle A, \alpha\rangle \ .$$

(3) (If $\mathcal{E}$ has kernel pairs of coequalizers and Γ preserves weak pullbacks) the equality relation $\Delta_{\langle A, \alpha\rangle}$ is the largest relation on $\langle A, \alpha\rangle$, i.e., the maximal element of $\mathsf{Rel}_{\mathcal{E}_\Gamma}(\langle A, \alpha\rangle, \langle A, \alpha\rangle)$.

PROOF. We prove that (1) and (2) are equivalent and that they imply (3). Then, we assume that Γ preserves weak pullbacks and prove that (3) implies (1).

- **(1) ⇒ (2):** Let $f, g : \langle B, \beta\rangle \to \langle A, \alpha\rangle$ be given and take the coequalizer of f and g (since $\mathcal{E}_\Gamma$ has all coequalizers). This coequalizer is again $\langle A, \alpha\rangle$, and so $f = g$.
- **(2) ⇒ (1):** Let

$$\langle B, \beta\rangle \underset{b_2}{\overset{b_1}{\rightrightarrows}} \langle A, \alpha\rangle \overset{q}{\twoheadrightarrow} \langle Q, \nu\rangle$$

  be a coequalizer diagram. Then $b_1 = b_2$, so $\langle Q, \nu\rangle \cong \langle A, \alpha\rangle$.
- **(2) ⇒ (3):** Let $\langle R, \rho\rangle$ be a relation on $\langle A, \alpha\rangle$, with projections r_1 and r_2. Then, $r_1 = r_2$, and so we have the factorization shown below.

$$\begin{tikzcd}
\langle R, \rho\rangle \arrow[rr, tail, "r_1 = r_2"] \arrow[dr, shift left, "r_1"] \arrow[dr, shift right, "r_2"'] & & \Delta_{\langle A, \alpha\rangle} \arrow[dl, shift right, "\mathsf{id}"'] \arrow[dl, shift left, "\mathsf{id}"] \\
& \langle A, \alpha\rangle &
\end{tikzcd}$$

- **(3) ⇒ (1):** Let $q : \langle A, \alpha\rangle \twoheadrightarrow \langle Q, \nu\rangle$ be a regular epi, and take the kernel pair of q in $\mathcal{E}$,

$$K \underset{k_2}{\overset{k_1}{\rightrightarrows}} A \overset{q}{\twoheadrightarrow} Q \ .$$

  Because Γ preserves weak pullbacks, there is a structure map for K,

$$\kappa : K \longrightarrow \Gamma K \, ,$$

making k_1 and k_2 homomorphisms. Because U reflects jointly monic families, $\langle K, \kappa\rangle$ is a relation on $\langle A, \alpha\rangle$, with projections k_1 and k_2. Since $\Delta_{\langle A,\alpha\rangle}$ is the largest relation, the following diagram commutes.

$$\begin{tikzcd}
\langle K, \kappa\rangle \arrow[rr, tail] \arrow[dr, shift left, "k_2"] \arrow[dr, shift right, "k_1"'] & & \Delta_{\langle A,\alpha\rangle} \arrow[dl, shift right, "\mathsf{id}"'] \arrow[dl, shift left, "\mathsf{id}"] \\
& A &
\end{tikzcd}$$

Hence, $k_1 = k_2$ and so, since every coequalizer is the coequalizer of its kernel pair, $\langle Q, \nu\rangle \cong \langle A, \alpha\rangle$.

□

Definition 1.5.26. We say that a coalgebra $\langle A, \alpha\rangle$ *satisfies the principle of coinduction* if $\Delta_{\langle A,\alpha\rangle}$ is the largest relation on $\langle A, \alpha\rangle$.

Corollary 1.5.27. *Any simple coalgebra satisfies the principle of coinduction. If $\mathcal{E}$ has kernel pairs of coequalizers and Γ preserves weak pullbacks, then any coalgebra satisfying the principle of coinduction is simple.*

We will return to the topic of coinduction in Section 2.6, where we will show how it leads to a proof principle for simple coalgebras.

1.5.4. The comparison map. Let $\Gamma:\mathcal{E}\to\mathcal{E}$ be given and suppose that $\mathcal{E}^\Gamma$ has an initial object, $\langle I, \iota\rangle$, and $\mathcal{E}_\Gamma$ a final object, $\langle F, \phi\rangle$. From Lambek's lemma, we know that the structure map ι is an isomorphism and the same holds for ϕ. Consequently, we can view the initial algebra as a coalgebra, namely, the coalgebra $\langle I, \iota^{-1}\rangle$. By finality, there is a unique map ! from $\langle I, \iota^{-1}\rangle$ to $\langle F, \phi\rangle$. That is, there is a unique map ! in $\mathcal{E}$ such that the diagram below commutes.

(5)
$$\begin{tikzcd}
\Gamma I \arrow[r, "\Gamma !"] & \Gamma F \\
I \arrow[u, "\iota^{-1}"] \arrow[r, "!"'] & F \arrow[u, "\phi"']
\end{tikzcd}$$

On the other hand, we can view the final coalgebra as an algebra, $\langle F, \phi^{-1}\rangle$. As such, there is a unique homomorphism from the initial algebra $\langle I, \iota\rangle$ to $\langle F, \phi^{-1}\rangle$. It is easy to see that this is simply two descriptions of the same map. The map ! in (5) clearly makes the diagram (6), below, commute.

(6)
$$\begin{tikzcd}
\Gamma I \arrow[r, "\Gamma !"] \arrow[d, "\iota"'] & \Gamma F \arrow[d, "\phi^{-1}"] \\
I \arrow[r, "!"'] & F
\end{tikzcd}$$

In fact, the map ! is just the unique map from the initial object to the final object in the category $\mathsf{Fix}(\Gamma)$. The point is that this map is a homomorphism in both relevant senses.

In the examples that we've seen thus far, this comparison map ! is precisely what one expects: it is an inclusion of the initial algebra into the final coalgebra. For instance, the initial algebra for the functor $\Gamma A = Z \times A + 1$ for a fixed X is the collection of finite streams over Z.

In [**Bar93**], Michael Barr shows that the initial algebra is often a dense subspace of the final coalgebra (under a natural topology), with the comparison map an inclusion. In [**Adá01**], Jiří Adámek extends these results.

CHAPTER 2

Constructions arising from a (co)monad

In this chapter, we focus on categories of (co)algebras which come with a left (right, resp.) adjoint to the forgetful functor. These categories are equivalent to categories of (co)algebras for a (co)monad, a stricter notion that categories of (co)algebras for an endofunctor. We begin the chapter with a review of (co)monads and their (co)algebras.

Following this, we introduce subcoalgebras. We view subcoalgebras as dual to quotients of algebras, and so take a subcoalgebra to be a regular subobject of the coalgebra. Theorems about subcoalgebras, then, are dual to theorems about quotients of algebras, or, equivalently when $\mathcal{E}$ is exact, theorems about congruences.

Given a Γ-coalgebra $\langle A, \alpha \rangle$, we introduce a modal operator $\Box$ on $\mathsf{Sub}(A)$, taking a subobject $P \leq A$ to the largest subcoalgebra $\langle B, \beta \rangle$ such that $B \leq P$. We show that $\Box$ is an **S4** modal operator. Furthermore, we discuss a left adjoint $\lhd$ taking P to the least subcoalgebra containing P. This closure operator exists if Γ preserves non-empty intersections.

We revisit the topic of limits in categories of coalgebras (and colimits in categories of algebras) and show that we may construct all limits (colimits, resp.) if the forgetful functor is comonadic (monadic, resp.). However, these constructed limits are not typically preserved by U.

We close the chapter by introducing the definition of bisimulations, which we take to be the image of a coalgebraic relation. This definition differs from the familiar definition in many texts, but we take our definition to be a reasonable expansion of the term for settings in which the axiom of choice is unavailable. We show, in fact, that the definition offered here coincides with the more traditional definition, given choice, and so feel that this generalization is suitable.

We discuss coinduction in terms of the introduced notion of bisimulation and also briefly generalize to n-simulations, to facilitate the development of the internal logic in Chapter 4.

2.1. (Co)monads and (co)algebras

A central notion in the study of universal algebras is the concept of a free algebra. Such algebras can be viewed as term algebras over a set of variables. Hence, from free algebras, one comes to a notion of equation (a pair of elements of the free algebra) and the definition of equation satisfaction. Categorically, such free algebras are easily understood in terms of adjoint functors. In particular, as we will see in Section 2.1.2, an algebra $\langle A, \alpha\rangle$ is free over X if $\langle A, \alpha\rangle \cong FX$, where F is the left adjoint to the forgetful functor.

Such adjoint functors give rise to monads in a natural way, which we discuss in the Appendix. One may ask whether every monad comes from a pair of adjoint functors. In fact, this is the case. Moreover, starting with a monad T, one can show that there are (at least) *two* methods of constructing an adjoint pair of functors that give rise to T. One method, the Kleisli construction, will not concern us much in what follows. Instead, we will focus on the Eilenberg-Moore construction. This construction considers the category of algebras for a monad T and shows that this category comes with a pair of adjoint functors $F \dashv U$ such that $T = UF$.

We begin by going into some detail on the definition of the category of algebras for a monad and sketch the proof of the Eilenberg-Moore theorem. This naturally leads into a discussion of (co)free (co)algebras in Section 2.1.2.

2.1.1. (Co)algebras for a (co)monad. In this section, we will define algebras for a monad and state the Eilenberg-Moore theorem. This theorem says that every monad arises as the monad for an adjunction. Moreover, every monad $\mathbb{T} = \langle T, \eta, \mu\rangle$ arises as the monad for an adjunction $F \dashv U$ where U is the forgetful functor for the category of $\mathbb{T}$-algebras. Here, however, we mean algebras for the *monad* $\mathbb{T}$. This is not the same as algebras for the endofunctor T — it is a narrower definition.

See Section A.5 for a brief review of monads.

In Section 2.1.2, we will discuss the situation in which the category of Γ algebras for an endofunctor Γ is equivalent to a category of algebras for a monad.

DEFINITION 2.1.1. Let $\mathbb{T} = \langle T, \eta, \mu\rangle$ be a monad over $\mathcal{E}$. A $\mathbb{T}$-*algebra* is an algebra $\langle A, \alpha\rangle$ for the endofunctor T such that the following diagrams commute.

$$\begin{tikzcd} T^2A \arrow[r, "\mu_A"] \arrow[d, "T\alpha"'] & TA \arrow[d, "\alpha"] \\ TA \arrow[r, "\alpha"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, equal] & TA \arrow[d, "\alpha"] \\ & A \end{tikzcd}$$

We refer to the commutativity of these diagrams as the *associativity* and *unit conditions for* $\mathbb{T}$-*algebras*. A $\mathbb{T}$-homomorphism is just a T-homomorphism in the sense of homomorphisms between algebras for an endofunctor (Definition 1.1.1). That is, a $\mathbb{T}$-homomorphism

$$f : \langle A, \alpha\rangle \longrightarrow \langle B, \beta\rangle$$

is a map $f\colon A \to B$ in $\mathcal{E}$ such that the diagram below commutes.

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\alpha"'] & TB \arrow[d, "\beta"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

The $\mathbb{T}$-algebras and their homomorphisms form a full subcategory of the $\mathcal{C}^T$, the category of algebras for the endofunctor T. We denote this category as $\mathcal{C}^{\mathbb{T}}$ (note the different font for the endofunctor T and the monad $\mathbb{T} = \langle T, \eta, \mu \rangle$).

Theorems 1.2.4, 1.2.7 and 1.2.13 hold in categories of coalgebras for a monad as well. That is, the forgetful functor creates limits, creates whatever colimits T preserves and $\mathcal{E}^{\mathbb{T}}$ inherits the regular epi-mono factorizations from an almost regular $\mathcal{E}$ if T also preserves regular epis. The first two facts can be found in [**Bor94**, Volume 2]. That $\mathcal{E}^{\mathbb{T}}$ has regular epi-mono factorizations is easily verified. In fact, $\mathcal{E}^{\mathbb{T}}$ is closed under quotients and subalgebras as a subcategory of $\mathcal{E}^T$ — indeed, it is a variety of $\mathcal{E}^T$ (see Section 3.2.1).

Dually, we define a $\mathbb{G}$-coalgebra for a comonad $\mathbb{G}$. This definition is a straightforward exercise in turning the arrows around in Definition 2.1.1, but we include it for reference.

Definition 2.1.2. Let $\mathbb{G} = \langle G, \varepsilon, \delta \rangle$ be a comonad over $\mathcal{E}$. A $\mathbb{G}$-*coalgebra* is a G-coalgebra $\langle A, \alpha \rangle$ such that the following diagrams commute.

$$\begin{tikzcd} G^2A & GA \arrow[l, "\delta_A"'] \\ GA \arrow[u, "G\alpha"] & A \arrow[l, "\alpha"] \arrow[u, "\alpha"'] \end{tikzcd} \qquad \begin{tikzcd} A & GA \arrow[l, "\varepsilon_A"'] \\ & A \arrow[ul, equal] \arrow[u, "\alpha"'] \end{tikzcd}$$

A $\mathbb{G}$-homomorphism is just a G-homomorphism between $\mathbb{G}$-coalgebras. The category of $\mathbb{G}$-coalgebras and their homomorphisms is denoted $\mathcal{E}_{\mathbb{G}}$. It is a full subcategory of the category $\mathcal{E}_G$ (indeed, a covariety).

The theorem below originally appeared in [**EM65**]. It can be found in any basic category theory text, including [**Bor94**, Volume 2] and [**BW85**]. We take it from the latter.

Theorem 2.1.3 (Eilenberg-Moore theorem). *Let $\mathbb{T} = \langle T, \eta, \mu \rangle$ be a monad over $\mathcal{E}$ and let $U\colon \mathcal{E}^{\mathbb{T}} \to \mathcal{E}$ be the evident forgetful functor. Then there is a functor $F\colon \mathcal{E} \to \mathcal{E}^{\mathbb{T}}$ such that $F \dashv U$ and $\mathbb{T}$ is the monad associated with the adjunction $F \dashv U$.*

Proof. We define $F\colon \mathcal{E} \to \mathcal{E}^{\mathbb{T}}$ on objects $C \in \mathcal{E}$ by

$$FC = \langle TC,\, \mu_C\colon T^2C \longrightarrow TC \rangle.$$

One must check that FC is a $\mathbb{T}$-algebra, i.e., that it satisfies the associativity and unit conditions for $\mathbb{T}$-algebras. This follows just from the associativity and (one of the) unit conditions for the monad $\mathbb{T}$ itself.

Let $C \in \mathcal{E}$ and $\langle A, \alpha\rangle \in \mathcal{E}^{\mathbb{T}}$. One must show

$$\mathsf{Hom}(\mathcal{E}, C)A \cong \mathsf{Hom}(\mathcal{E}^{\mathbb{T}}, FC)\langle A, \alpha\rangle.$$

The isomorphism takes a map $f : C \to A$ to

$$\alpha \circ Tf : FC \longrightarrow \langle A, \alpha\rangle.$$

The inverse takes a homomorphism $g : FC \to \langle A, \alpha\rangle$ to

$$g \circ \eta_C : C \longrightarrow A.$$

Clearly, we have that $T = UF$, as desired. One must check that the unit of the adjunction $F \dashv U$ is η, the unit of $\mathbb{T}$, and that the multiplication μ of $\mathbb{T}$, is given by $U\varepsilon_F$, where ε is the counit of the adjunction. This is easy.

It is worth noting that the counit of the adjunction arises naturally enough: If $\langle A, \alpha\rangle$ is a $\mathbb{T}$-algebra, then $\varepsilon_{\langle A, \alpha\rangle} = \alpha$. ☐

So, given any monad $\mathbb{T}$, we can "factor" $\mathbb{T}$ into an adjoint pair via the Eilenberg-Moore construction. This factorization is not unique, however. Indeed, every monad has at least one other factorization: the factorization given by Kleisli in [**Kle65**]. However, the Eilenberg-Moore factorization is distinguished: It is final among all such factorizations[1]. We state the theorem more precisely here, but it will not play a significant role in this thesis.

THEOREM 2.1.4. *Let* $\mathbb{T} = \langle T, \eta, \mu\rangle$ *and*

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, bend left, "R"] \end{tikzcd}$$

be given such that

- $T = R \circ L$
- *The unit of the adjunction* $L \dashv R$ *is* η*, the unit of the monad.*
- *The multiplication of the monad,* μ*, is equal to* $R\varepsilon_L$*, where* ε *is the counit of the adjunction* $R \circ L$*.*

[1]The Kleisli construction is initial among all such factorizations. See [**Bor94**, Volume 2] or [**BW85**].

Then, there is a unique $J:\mathcal{D}\to\mathcal{E}^{\mathbb{T}}$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, bend left, "J"] \arrow[dr, bend right, "R"'] & & \mathcal{E}^{\mathbb{T}} \arrow[dl, bend right, "U"'] \\
& \mathcal{E} \arrow[ul, bend right, "L"'] \arrow[ur, bend right, "F"'] &
\end{tikzcd}$$

The Eilenberg-Moore construction dualizes in a natural way. Given a comonad $\mathbb{G} = \langle G, \varepsilon, \delta\rangle$ over $\mathcal{E}$, the forgetful functor

$$U:\mathcal{E}_{\mathbb{G}} \longrightarrow \mathcal{E}$$

has a right adjoint,

$$H:\mathcal{E} \longrightarrow \mathcal{E}_{\mathbb{G}}\,,$$

such that the comonad $\mathbb{G}$ is induced by the adjunction $U \dashv H$. The functor H takes an object $C \in \mathcal{E}$ to the coalgebra $\langle GC, \delta_C\rangle$.

2.1.2. Free algebras. A basic notion in the theory of universal algebras is that of the free algebra over a set of variables. Let Σ be a signature and X a set of variables. The free Σ-algebra over X, denoted FX, can be described informally as the collection of all terms that can be constructed from the variables of X using the function symbols of Σ. This informal description can be stated more precisely in terms of least fixed points, but we do not take these descriptions to be the *definition* of a free algebra over X. Instead, the property of freeness is defined in terms of homomorphic extensions of maps.

Specifically, the property of freeness says: for every Σ-algebra $\langle A, \alpha\rangle$ and every assignment σ of the variables of X to the carrier A, there is a unique homomorphism $\widetilde{\sigma}:FX\to\langle A, \alpha\rangle$ extending the assignment σ. An assignment of the variables of X to A is just a map $\sigma:X\to A$. Thus, the defining property of FX can be stated: there is a map

$$\eta_X:X \longrightarrow UFX$$

(called *the insertion of generators*) such that, for every

$$\sigma:X \longrightarrow A,$$

there is a unique

$$\widetilde{\sigma}:FX \longrightarrow \langle A, \alpha\rangle$$

making the following diagram commutes:

$$\begin{tikzcd}
& UFX \arrow[d, "\widetilde{\sigma}"] \\
X \arrow[ur, "\eta_X"] \arrow[r, "\sigma"'] & A
\end{tikzcd}$$

This condition should look familiar. If, for every $X \in \mathbf{Set}$, there is a free algebra over X, then the operator F taking each X to its free algebra extends to a functor which is left adjoint to the forgetful functor U.

This allows us to state quite abstractly what it means for a Γ-algebra to be a free algebra over some object X. Namely, $\langle K, \kappa\rangle$ is *free over* X just in case there is a left adjoint $F: \mathcal{E} \to \mathcal{E}^{\Gamma}$ to the forgetful functor U and $\langle K, \kappa\rangle \cong FX$. Notice that an initial Γ-algebra is free over the initial object of $\mathcal{E}$, if it exists.

The universal mapping property of the free algebra gives another description of it. Let $FX = \langle K, \kappa\rangle$ be the free Γ-algebra over X. We have a pair of maps, then,

$$X \xrightarrow{\eta_X} K \xleftarrow{\kappa} \Gamma K ,$$

and so we can consider the $X+\Gamma$-coalgebra, $\langle K, [\eta_X, \kappa]\rangle$.

By the adjunction $F \dashv U$, we have, for all Γ-algebras $\langle A, \alpha\rangle$ and maps $\sigma : X \to A$, there is a unique Γ-homomorphism $\widetilde{\sigma}$ such that $\widetilde{\sigma} \circ \eta_X = \sigma$. Any such Γ-algebra $\langle A, \alpha\rangle$ and assignment σ corresponds to an $X+\Gamma$-algebra, namely $\langle A, [\sigma, \alpha]\rangle$. Furthermore, by the conditions of the adjunction, the diagram below, commutes.

$$\begin{tikzcd}
 & K \arrow[d, "\widetilde{\sigma}"] & \Gamma K \arrow[l, "\kappa"'] \arrow[d, "\Gamma\widetilde{\sigma}"] \\
X \arrow[ur, "\eta_X"] \arrow[r, "\sigma"'] & A & \Gamma A \arrow[l, "\alpha"]
\end{tikzcd}$$

But, this is exactly the condition needed to show that $\widetilde{\sigma}$ is an $X+\Gamma$-homomorphism, i.e., that the following diagram commutes.

$$\begin{tikzcd}
X + \Gamma K \arrow[r, "\mathsf{id}_X + \Gamma\widetilde{\sigma}"] \arrow[d, "{[\eta_X, \kappa]}"'] & X + \Gamma A \arrow[d, "{[\sigma, \alpha]}"] \\
K \arrow[r, "\widetilde{\sigma}"'] & A
\end{tikzcd}$$

Thus, we see that $\langle K, \kappa\rangle$ satisfies the following condition: For every $X+\Gamma$-algebra $\langle A, [\sigma, \alpha]\rangle$, there is a unique $X+\Gamma$-homomorphism $\langle K, [\eta_X, \kappa]\rangle \longrightarrow \langle A, [\sigma, \alpha]\rangle$. In other words, $\langle K, [\eta_X, \kappa]\rangle$ *is the initial* $X+\Gamma$*-algebra.*

This observation leads to an alternative definition of free algebra over X, one that does not require that every object of $\mathcal{E}$ has a free algebra. Namely, we say that $\langle K, \kappa\rangle$ is a free Γ-algebra over X just in case there is a map $f: X \to K$ such that $\langle K, [f, \kappa]\rangle$ is an initial $X+\Gamma$ algebra. We have, then, the following fact:

THEOREM 2.1.5. *Let $\mathcal{E}$ have binary coproducts, $\Gamma: \mathcal{E} \to \mathcal{E}$ be given and $U: \mathcal{E}^{\Gamma} \to \mathcal{E}$ be the forgetful functor. Then U has a left adjoint F iff, for each $X \in \mathcal{E}$, the initial $X+\Gamma$-algebra exists.*

As an immediate corollary, we have

COROLLARY 2.1.6. *Let* $\Gamma : \mathcal{E} \to \mathcal{E}$ *be given and let* $F : \mathcal{E} \to \mathcal{E}^\Gamma$ *be the left adjoint to the forgetful functor* $U : \mathcal{E}^\Gamma \to \mathcal{E}$*, with* η *the unit of the adjunction. Then, for every* $X \in \mathcal{E}$*,*

$$[\eta_X, \kappa] : X + \Gamma UFX \longrightarrow UFX$$

is an isomorphism, where $\kappa : \Gamma UFX \to UFX$ *is the structure map of the algebra* FX *(so* $FX = \langle UFX, \kappa \rangle$*).*

PROOF. Lambek's lemma (Lemma 1.5.1). □

Thus, the existence of a free functor F depends on whether an initial $X + \Gamma$-algebra exists for every $X \in \mathcal{E}$. So, one can use existence theorems for initial algebras to prove that U has a left adjoint. For instance, if Γ is co-continuous, then, for every X, the functor $X + \Gamma$ is also co-continuous and so has an initial algebra (from a well-known fixed point theorem, generalized in [**Bar92**]). For the most part, we will not be concerned here with the question of the existence of a functor F, any more than we are concerned with the existence theorems for initial algebras and final coalgebras.

We can apply the results of Section 1.5.2 to free algebras. Since a free Γ-algebra over X is an initial $X + \Gamma$-algebra, it comes with the proof principles common to all initial algebras: induction and recursion. We have seen the principle of recursion. It is the principle that, for every Γ-algebra $\langle A, \alpha \rangle$ and every map $\sigma : X \to A$, there is a unique homomorphism $\widetilde{\sigma} : FX \to \langle A, \alpha \rangle$ extending σ. This gives a nice description of ε, the counit of the adjunction $F \dashv U$, namely, $\varepsilon_{\langle X, \xi \rangle}$ is the extension of the assignment $\mathsf{id}_X : X \to X$.

The principle of induction for free algebras should be familiar as well. This principle commonly occurs in proof theory, for instance — it is the principle of structural induction for terms. After all, the term algebras for a language are just $\mathbb{P}$-algebras for some polynomial functor $\mathbb{P}$. If $\mathbb{P}$ is a polynomial, then $X + \mathbb{P}$ is also a polynomial. Thus, structural induction for terms over a set of variables is just a special case of Example 1.5.11.

Induction for free algebras for other functors is similar. It states that, for each property P, if P holds of the elements of X, and if P is preserved under "term formation" (whatever that means for the functor at hand), then P holds for all of FX.

EXAMPLE 2.1.7. Consider the **Set**-functor $\Gamma A = Z \times A + 1$, from Example 1.1.6. The forgetful functor $U : \mathbf{Set}^\Gamma \to \mathbf{Set}$ has a left adjoint, $F : \mathbf{Set} \to \mathbf{Set}^\Gamma$. The functor F takes a set X to the initial $X + Z \times - + 1$ algebra. We can understand this object as the initial algebra for a polynomial functor. Hence, we can think of it as a collection of terms for a signature (Example 1.1.5). The signature includes a constant for each

$x \in X$ and also a constant $*$ for the unique element of 1. Also, for each $z \in Z$, we have a unary function symbol $z^{(1)}$. Thus, the free Γ-algebras are easily understood.

But, this characterization isn't very useful for the interpretation of $\mathbf{Set}^\Gamma$ we've chosen. We've said that the initial algebra for $\mathbf{Set}^\Gamma$ is the collection of finite streams over Z, denoted $Z^{<\omega}$. So, we would like to describe the free algebras in these terms as well. Of course, the initial algebra itself is a free algebra — it is the free algebra over the empty set, $F0$. So, we also want our description of free algebras to coincide with our description of the initial algebra.

Let $X \in \mathbf{Set}$. We consider the elements of UFX as finite streams over Z again, with one important difference. In the initial Γ-algebra, there is a single object that represents an empty stream, which we denote $()$. In the Γ-algebra FX, there are many "empty streams". In addition to $()$, we have an empty stream for each $x \in X$. Let

$$[\mathsf{push}_X, ()_X]: Z \times UFX + 1 \longrightarrow UFX$$

be the structure map for FX and η the unit of the adjunction $F \dashv U$. The map

$$[\eta_X, ()]: X + 1 \longrightarrow UFX$$

picks out these empty streams, while the map push_X constructs a new stream from an element of Z together with an element of UFX.

More concretely, the free Γ-algebra over X is given by

$$UFX = (X + 1) \times Z^{<\omega}.$$

The $X + 1$ component denotes the "type" of the end of the stream. The structure map

$$[\mathsf{push}_X, ()_X]: Z \times UFX + 1 \longrightarrow UFX$$

is defined by

$$()_X = \langle *, ()\rangle,$$
$$\mathsf{push}_X(z, \langle a, \sigma\rangle) = \langle a, \mathsf{push}(z, \sigma)\rangle,$$

where $z \in Z$, $a \in X + 1$ and $\sigma \in Z^{<\omega}$. The functions $()$ and push here were defined for the initial algebra

$$\langle Z^{<\omega}, [\mathsf{push}, ()]\rangle$$

in Example 1.5.6.

2.1.3. Monadicity. We now return to the topic of algebras for a monad and show how it relates to free algebras for an endofunctor: Specifically, if the forgetful functor $U: \mathcal{E}^\Gamma \to \mathcal{E}$ has a left adjoint F, then $\mathcal{E}^\Gamma$ is isomorphic to $\mathcal{E}^{\mathbb{T}}$, where $\mathbb{T}$ is the triple induced by the adjunction $F \dashv U$. Moreover, the isomorphism commutes with the respective forgetful functors. Thus, U is *monadic* in the sense below.

Definition 2.1.8. Let $G:\mathcal{C}\to\mathcal{D}$ be given. We say that G is *monadic* if there is a monad $\mathbb{T}$ on $\mathcal{D}$ and an equivalence of categories $J:\mathcal{C}\to\mathcal{D}^{\mathbb{T}}$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left, "J"] \arrow[dr, "G"'] & & \mathcal{D}^{\mathbb{T}} \arrow[dl, "U"] \\
& \mathcal{D} &
\end{tikzcd}$$

The functor U, above, is the forgetful functor for the category of algebras for the monad $\mathbb{T}$.

One can learn about monadic functors in the standard category theory texts. This definition and Theorem 2.1.10, below, come from [**Bor94**, Volume 2]. They can also be found in [**BW85**]. Before stating Beck's theorem, we must have a definition, also from [**Bor94**, Volume 2].

Definition 2.1.9. A diagram of the form

$$\begin{tikzcd}
A \arrow[r, shift left, "a_1"] \arrow[r, shift right, "a_2"'] & B \arrow[l, bend right, "f"'] \arrow[r, bend right, "q"'] & Q \arrow[l, bend right, "g"']
\end{tikzcd}$$

is a *split coequalizer* if the following hold:

$$\begin{aligned}
q \circ a_1 &= q \circ a_2 \\
q \circ g &= \mathsf{id}_Q \\
a_1 \circ f &= \mathsf{id}_B \\
a_2 \circ f &= g \circ q
\end{aligned}$$

It is easy to check that split coequalizers are indeed coequalizers and moreover are absolute (preserved by every functor). Split coequalizers naturally arise in the context of algebras for an monad $\mathbb{T} = \langle T, \eta, \mu\rangle$ since, for any $\mathbb{T}$-coalgebra $\langle A, \alpha\rangle$, the diagram below is a split coequalizer.

$$\begin{tikzcd}
T^2A \arrow[r, shift left, "\mu_A"] \arrow[r, shift right, "T\alpha"'] & TA \arrow[l, bend right, "\eta_{TA}"'] \arrow[r, bend right, "\alpha"'] & Q \arrow[l, bend right, "\eta_A"']
\end{tikzcd}$$

It is this fact which is crucial in the characterization of monadic functors, first due to J. M. Beck [**Bec67**].

Theorem 2.1.10 (Beck's theorem). *Let $G:\mathcal{D}\to\mathcal{C}$ be given. The following are equivalent.*

(1) G is monadic.
(2) (a) G has a left adjoint.
(b) G reflects isomorphisms.

(c) For any pair

$$A \overset{f}{\underset{g}{\rightrightarrows}} B$$

such that Gf and Gg have a split coequalizer in $\mathcal{C}$, f and g have a coequalizer in $\mathcal{D}$ which is preserved by G.

COROLLARY 2.1.11. *Let $\Gamma : \mathcal{E} \to \mathcal{E}$ be given. The forgetful functor $U : \mathcal{E}^\Gamma \to \mathcal{E}$ has a left adjoint iff U is monadic.*

PROOF. Let Γ be given. Because U is faithful, it reflects isomorphisms. If homomorphisms $f, g : \langle A, \alpha\rangle \to \langle B, \beta\rangle$ have a split coequalizer in $\mathcal{E}$, then Γ preserves the split coequalizer (since it is an absolute coequalizer). Hence, U reflects and preserves the split coequalizer. Thus, applying Theorem 2.1.10 completes the proof. $\square$

In [**AP01**], a functor Γ is called a *varietor* if the forgetful functor $U : \mathcal{E}^\Gamma \to \mathcal{E}$ is monadic.

We can strengthen the results of this corollary. First, we put the corollary in the context of results from Section 2.1.2. Let $\Gamma : \mathcal{E} \to \mathcal{E}$ be given and suppose that $U : \mathcal{E}^\Gamma \to \mathcal{E}$ has a left adjoint $F : \mathcal{E} \to \mathcal{E}^\Gamma$. Let $\mathbb{T} = \langle T = UF, \eta, U\varepsilon_F\rangle$ be the monad induced by the adjunction $F \dashv U$, and let

$$\begin{tikzcd} \mathcal{E} \arrow[r, bend left, "F^{\mathbb{T}}"] \arrow[r, phantom, "\perp"] & \mathcal{E}^{\mathbb{T}} \arrow[l, bend left, "U^{\mathbb{T}}"] \end{tikzcd}$$

be the adjoint functors given in the Eilenberg-Moore theorem (Theorem 2.1.3). One can show that there is an isomorphism (rather than a mere equivalence) $\mathcal{E}^\Gamma \cong \mathcal{E}^{\mathbb{T}}$ that commutes with the forgetful functors.

A nice presentation of this fact is given in Daniele Turi's dissertation [**Tur96**]. The reader should look there for the details, but it is worth describing the action of J and its inverse. We map a Γ-algebra,

$$\langle A, \alpha : \Gamma A \longrightarrow A\rangle,$$

to the $\mathbb{T}$-algebra

$$\langle A, U\varepsilon_\alpha : TA \longrightarrow A\rangle.$$

On the other hand, suppose we start with a $\mathbb{T}$-algebra,

$$\langle C, \gamma : TC \longrightarrow C\rangle.$$

Recall that we have an isomorphism $C + \Gamma TC \rightarrowtriangle TC$ (Corollary 2.1.6). The left component $C \to TC$ is the unit of the monad (and of the adjunction $F \dashv U$). Call the right component θ_C. Then, we map $\langle C, \gamma\rangle$ to the Γ-algebra with structure map

$$\Gamma C \xrightarrow{\Gamma\eta_C} \Gamma TC \xrightarrow{\theta_C} TC \xrightarrow{\gamma} C$$

We omit the proof that these operators extend to functors that are inverses of each other.

This concludes our discussion of free algebras and the associated category of algebras for a monad. We have covered this well-traveled ground in order to consider the dual case. In Section 2.1.4, we will put the algebraic theorems to work in order to learn about cofree coalgebras.

2.1.4. Cofree coalgebras. An early discussion of cofree coalgebras occurs in [**Rut96**]. There, Rutten gives the now familiar discussion of cofreeness in terms of colorings. This interpretation of cofreeness arises naturally from dualizing the work in Section 2.1.2. We follow this approach.

Previously, we saw that a Γ-algebra $\langle A,\, \alpha\rangle$ is free over X just in case there is a map $\eta_X : X \to A$ such that $\langle A,\, [\eta_X, \alpha]\rangle$ is the initial $X + \Gamma$-algebra. We dualize this observation to define cofreeness.

Definition 2.1.12. Let $\Gamma : \mathcal{E} \to \mathcal{E}$ be given and let C be an object of $\mathcal{E}$. A Γ-coalgebra $\langle A,\, \alpha\rangle$ is *cofree over* C just in case there is a map $\varepsilon_C : A \to C$ such that the $C \times \Gamma$-coalgebra $\langle A,\, \langle \varepsilon_C,\, \alpha\rangle\rangle$ is final (in the category $\mathcal{E}_{C\times\Gamma}$).

Let $\langle A,\, \alpha\rangle$ be cofree over C and let $\langle B,\, \beta\rangle$ be a Γ-coalgebra. Then, for any $p : B \to C$, we have a $C \times \Gamma$-coalgebra, namely, $\langle B,\, \langle p,\, \beta\rangle\rangle$. Thus, there is a unique map $f : B \to A$ such that the diagram below commutes.

$$\begin{tikzcd}
 & A \arrow[r, "\alpha"] \arrow[dl, "\varepsilon_C"'] & \Gamma A \\
C & B \arrow[l, "\beta"] \arrow[u, "f"'] \arrow[r, "p"'] & \Gamma B. \arrow[u, "\Gamma f"']
\end{tikzcd}$$

We understood free algebras over X by considering X to be a set of variables. The free algebra over X, then, was the collection of Γ-terms over a set of variables. When considering cofree coalgebras over C, we imagine C to be a set of colors. We interpret maps $p : B \to C$ as *colorings of the elements of* B *by the colors* C (i.e., as a C-coloring of B). To each element of B, the coloring p assigns a color from C. The map ε_C is also a coloring: It colors the elements of the cofree coalgebra $\langle A,\, \alpha\rangle$. Thus, we can state the principal of cofreeness as follows: $\langle A,\, \alpha\rangle$ is cofree over C iff there is a C-coloring ε_C of A such that, for every Γ-coalgebra $\langle B,\, \beta\rangle$ and C-coloring p of B, there is a unique homomorphism $f : \langle A,\, \alpha\rangle \to \langle B,\, \beta\rangle$ "consistent" with the coloring p. By consistent, we simply mean that the following diagram commutes, so that elements of B are mapped to elements of A of the same color (under p and ε_C, respectively).

$$\begin{tikzcd} & A \arrow[d, "\varepsilon_C"] \\ B \arrow[ur, "f"] \arrow[r, "p"'] & C \end{tikzcd}$$

Example 2.1.13. Consider the **Set**-functor $\Gamma A = Z \times A$ from Example 1.1.7 and let C be a set. Then, $\langle A, \alpha \rangle$ is cofree over C just in case there is a coloring $\varepsilon_C : A \to C$ such that $\langle A, \langle \varepsilon_C, \alpha \rangle \rangle$ is the final $C \times \Gamma$-coalgebra.

A $C \times \Gamma$-coalgebra is just a coalgebra for the functor

$$A \mapsto C \times Z \times A.$$

Thus, the final $C \times \Gamma$-coalgebra is the collection of all streams over $C \times Z$. Therefore, the cofree coalgebra over C exists and is given by $(C \times Z)^\omega$, with the evident structure map and coloring (counit).

Let $\mathcal{E}$ and Γ be given and suppose that, for every $C \in \mathcal{E}$, there is a cofree coalgebra over C. Then, there is a

$$H : \mathcal{E} \longrightarrow \mathcal{E}_\Gamma$$

such that H is right adjoint to the forgetful functor $U : \mathcal{E}_\Gamma \to \mathcal{E}$. Namely, we take HC to be the cofree Γ-coalgebra over C. Indeed, the principal of cofreeness leads directly to the adjunction conditions: For every

$$p : U\langle B, \beta \rangle \longrightarrow C,$$

there exists a unique

$$\widetilde{p} : \langle B, \beta \rangle \longrightarrow HC$$

such that the following diagram commutes.

$$\begin{tikzcd} & UHC \arrow[d, "\varepsilon_C"] \\ B \arrow[ur, "U\widetilde{p}"] \arrow[r, "p"'] & C \end{tikzcd}$$

Notice that the C-coloring ε_C of the cofree coalgebra HC is the component at C of the counit of the adjunction $U \dashv H$. This is analogous to the result in Section 2.1.2 that the insertion of variables arose from the unit of the adjunction $F \dashv U$.

We can also dualize the monadicity results from Section 2.1.3. Accordingly, we define comonadic functor below and show that, if $U : \mathcal{E}_\Gamma \to \mathcal{E}$ has a right adjoint, then U is comonadic. We do this directly, without discussing the dual of Beck's theorem, since split equalizers do not play a significant role either in the literature or in the remainder of this thesis.

DEFINITION 2.1.14. Let $K\colon\mathcal{C}\to\mathcal{D}$ be given. We say that K is *comonadic* if there is a comonad $\mathbb{G}$ on $\mathcal{D}$ and an equivalence of categories $J\colon\mathcal{C}\to\mathcal{D}_{\mathbb{G}}$ such that the following diagram commutes.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, bend left, "J"] \arrow[dr, "K"'] & & \mathcal{D}_{\mathbb{G}} \arrow[dl, "U_{\mathbb{G}}"] \\ & \mathcal{D} & \end{tikzcd}$$

The functor $U_{\mathbb{G}}$, above, is the forgetful functor for the category of coalgebras for the comonad $\mathbb{G}$.

THEOREM 2.1.15. *Let $\mathcal{E}$ and Γ be given such that the forgetful functor $U\colon\mathcal{E}_\Gamma\to\mathcal{E}$ has a right adjoint $H\colon\mathcal{E}\to\mathcal{E}_\Gamma$. Then U is comonadic.*

PROOF. We simply sketch the proof here, since the dual construction was discussed in Section 2.1.3. The category $\mathcal{E}_\Gamma$ is isomorphic to the category $\mathcal{E}_{\mathbb{G}}$ of coalgebras for the comonad

$$\mathbb{G} = \langle G = UH,\ \varepsilon,\ U\eta_H\rangle$$

where ε and η are the counit and unit, respectively, of the adjoint $U \dashv H$. The isomorphism takes a Γ-coalgebra $\langle A,\ \alpha\rangle$ to the $\mathbb{G}$-coalgebra

$$\langle A,\ U\eta_\alpha\colon A\longrightarrow GA\rangle.$$

The inverse takes a $\mathbb{G}$-coalgebra $\langle C,\ \gamma\rangle$ to the Γ-coalgebra

$$\langle C,\ \gamma\circ\xi_C\circ\Gamma\varepsilon_C\rangle,$$

where $\xi_C\colon GC\to\Gamma GC$ arises from the isomorphism

$$GC \xrightarrow{\langle\varepsilon_C,\,\xi_C\rangle} C\times GC.$$

□

In [**AP01**], a functor Γ is called a *covarietor* just in case the coalgebraic forgetful functor $U\colon\mathcal{E}_\Gamma\to\mathcal{E}$ is comonadic (equivalently, has a right adjoint).

This next theorem is dual to the well-known fact that, if $\Gamma\colon\mathbf{Set}\to\mathbf{Set}$ is a varietor, then an algebra $\langle A,\ \alpha\rangle$is the quotient of FA.

THEOREM 2.1.16. *Let $\mathcal{E}$ be almost co-regular and let Γ preserve regular monos. Suppose further that $U\colon\mathcal{E}_\Gamma\to\mathcal{E}$ has a right adjoint H. Then, each Γ-coalgebra $\langle A,\ \alpha\rangle$ is a regular subcoalgebra of HA.*

PROOF. Let η and ε be the unit and counit, resp., of the adjunction $U \dashv H$. Then, it is a basic fact of adjunctions that $\varepsilon_U\circ U\eta = \mathsf{id}_{\mathcal{E}}$ (see, for instance, [**Bor94**, Chapter 4, Volume 2] or any other introduction of monads). Thus,

$$\varepsilon_A\circ U\eta_{\langle A,\,\alpha\rangle} = \mathsf{id}_A$$

and so $U\eta_{\langle A,\,\alpha\rangle}$ is a regular mono. Since U reflects regular monos (by the dual of Corollary 1.2.15), we have

$$\eta_{\langle A,\,\alpha\rangle} : \langle A,\, \alpha\rangle \rightarrowtail HA.$$

$\square$

REMARK 2.1.17. The dual of this is worth mentioning: If $\mathcal{E}$ satisfies the conditions of Corollary 1.2.15, and the algebraic forgetful functor $U : \mathcal{E}^{\Gamma} \to \mathcal{E}$ is monadic, then each algebra $\langle A,\, \alpha\rangle$ is a quotient of the free algebra FA.

2.1.5. Covarietors and inheritance. In the remainder of this section, we sketch an application of covarietors which has not, apparently, been explored in the literature. As is well-known, categories of coalgebras over **Set** can be used to model objects in an object oriented programming language (at least certain objects — we ignore here constructors and other complications found in [**PZ01**]). Typically, the functors one uses for the categories of coalgebras are polynomial functors and hence are covarietors.

For our purposes at present, a class specification consists of a list of methods (together with their signatures). For example, consider the specification below:

begin `Counter`
  operations
    $\mathsf{inc} : X \to X$
    $\mathsf{val} : X \to \mathbb{N}$
end `Counter`

This specification describes a class `Counter` with two methods, inc and val. It should be clear that any $(- \times \mathbb{N})$-coalgebra provides a set of such `Counter`s, and so we call such coalgebras interpretations of the specification `Counter`. (We do not intend here to give a rigorous presentation of coalgebraic semantics for class specification, but rather a reasonable sketch of this topic. See [**RTJ01**] for a development of this topic.) Of course, most $(- \times \mathbb{N})$-coalgebras do not behave like a proper counter — certainly, we have not required that $\mathsf{val}(\mathsf{inc}(x)) = \mathsf{val}(x) + 1$ here. The name `Counter` is meant to be suggestive, but for the purposes of this example, a specification merely gives the signatures of the methods, without any assertions about the behavior of these methods. See, however, Example 3.6.16 for a discussion of such assertions and their relation to coequations.

Often, given such a class specification, one extends the specification to a new class, which is augmented with additional methods. For instance, given the specification of `Counter` above, we may wish to specify a counter which comes with a decrement method (in addition to the increment and value methods of `Counter`). Thus, we may wish to give a specification as shown below:

begin `DecCounter` **extends** `Counter`
  operations
    $\mathsf{dec}: X \to X + 1$
end `DecCounter`

We could model `DecCounter` by $(- \times \mathbb{N} \times (- + 1))$-coalgebras, and this is what is typically done. However, there is a sense in which this interpretation neglects the relation between `DecCounter` and `Counter`. Since `DecCounter` arises by adding methods to `Counter`, it seems natural to consider interpretations of `DecCounter` to be coalgebras over $\mathbf{Set}_{(-\times\mathbb{N})}$, the category of interpretations of `Counter`. Thus, we would like to find a functor

$$\Delta : \mathbf{Set}_{(-\times\mathbb{N})} \longrightarrow \mathbf{Set}_{(-\times\mathbb{N})}$$

such that $(\mathbf{Set}_{(-\times\mathbb{N})})_\Delta \equiv \mathbf{Set}_{(-\times\mathbb{N}\times(-+1))}$.

One would be tempted to take Δ to be the obvious functor, $\Delta X = X + 1$, since we are adding a method of type $X \to X + 1$. However, this will not work, since we do not expect the structure map dec to be a $(- \times \mathbb{N})$-homomorphism. Instead, it suffices to take Δ to be the composite $H \circ (- + 1) \circ U$, as the following theorem shows.

THEOREM 2.1.18. *Let $\mathcal{E}$ be a category with binary products, $\Gamma : \mathcal{E} \to \mathcal{E}$ be a covarietor and $\Delta : \mathcal{E} \to \mathcal{E}$ any endofunctor. Then*

$$\mathcal{E}_{\Gamma\times\Delta} \cong (\mathcal{E}_\Gamma)_{H\Delta U}.$$

PROOF. Let $\langle A, \alpha : A \to (\Gamma \times \Delta)A\rangle$ be a $\Gamma \times \Delta$-coalgebra. Then $\langle A, \pi_1\alpha\rangle$ is a Γ-algebra. Let

$$\alpha' : \langle A, \pi_1\alpha\rangle \longrightarrow H\Delta U\langle A, \pi_1\alpha\rangle$$

be the adjoint transpose of $\pi_2\alpha : A \to \Delta A$. Then $\langle\langle A, \pi_1\alpha\rangle, \alpha'\rangle$ is an $H\Delta U$-coalgebra (over $\mathcal{E}_\Gamma$). It is easy to check that this construction is functorial and yields the isomorphism desired. □

2.2. Subcoalgebras

In Sections 1.3 and 1.4, we introduces subalgebras and congruences. A subalgebra of $\langle A, \alpha\rangle$ is a subobject of A which is closed under the structure map α. A precongruence on $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ is a relation on A and B which is similarly closed under the operations of α and β. The attractiveness of these definitions come from the view that subobjects and relations on $\mathcal{E}$ are familiar concepts, so we focus attention on subobjects and relations in $\mathcal{E}^\Gamma$ which are also subobjects and relations in $\mathcal{E}$ (that is, are mapped to subobjects and relations by the forgetful functors U_α and $U_{\alpha,\beta}$, respectively).

Our definition of the corresponding notions, subcoalgebra and bisimulation, will be similarly motivated. A subcoalgebra of $\langle A, \alpha\rangle$ is a subobject of $\langle A, \alpha\rangle$ which is preserved by the forgetful functor. A bisimulation R on $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ is a relation on A and B — so, it is a relation in a familiar sense. However, the definition is a bit more complicated than the definition of a pre-congruence. We will not require that

$$R = U_{\alpha\times\beta}(\langle S, \sigma\rangle)$$

for some relation $\langle S, \sigma\rangle$ on $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$. Instead, we require that R is the image of some $U_{\alpha\times\beta}(\langle S, \sigma\rangle)$. We discuss bisimulations in detail in Section 2.5.

For subcoalgebras, we have a separate motivation which determines our definition. In categories of algebras, regular epis play a central role in the development of the theory. Indeed, the correspondence between regular epis and congruences can be viewed as a key reason that congruences are an important concept for categories of algebras. As we will see in Chapter 3, when reasoning about congruences (in this case, deductively closed sets of equations), it is convenient to reason about their quotients and translate the results into theorems about congruences. If congruences play a more central role in the theorems than quotients, it is because relations seem a more familiar concept than their coequalizers.

If we take the straightforward approach and define a subcoalgebra as a subobject in $\mathsf{Sub}(\langle A, \alpha\rangle)$ which is preserved by U_α, then we lose the structural advantage that regular epis have in categories of algebras over epis in general. Just as regular epis[2] are central in $\mathcal{E}^\Gamma$, one expects that their dual, regular monos, will play a central role in the dual category, $\mathcal{E}_\Gamma$. Thus, we offer the following definition.

Definition 2.2.1. Let $\langle A, \alpha\rangle$ be a Γ-coalgebra. A *subcoalgebra of* $\langle A, \alpha\rangle$ is a Γ-coalgebra $\langle B, \beta\rangle$ together with a regular mono homomorphism

$$i : \langle B, \beta\rangle \rightarrowtail \langle A, \alpha\rangle\,.$$

The category of (equivalence classes of) subcoalgebras of $\langle A, \alpha\rangle$ is denoted

$$\mathsf{SubCoalg}(\langle A, \alpha\rangle).$$

Example 2.2.2. Let $\langle A, \mathcal{O}_A\rangle$ be a topological space and $\langle A, \alpha\rangle$ the associated $\mathcal{F}$-coalgebra (see Example 1.1.12). Then $\langle B, \beta\rangle$ is a subcoalgebra of $\langle A, \alpha\rangle$ iff B is (isomorphic to) an open subset of $\langle A, \alpha\rangle$ and β is the neighborhood filter on the subspace $\langle B, \mathcal{O}_B\rangle$.

Throughout this section, we assume that Γ preserves regular monos. Thus, if $\langle B, \beta\rangle$ is a subcoalgebra of $\langle A, \alpha\rangle$, then B is a regular subobject of A, so subcoalgebras are regular subobjects in $\mathcal{E}$. In a more general setting, we would make a

[2]In fact, it would be just as well to work with strong epis and monos, and alter the theorems accordingly, but we would lose the connection between quotients and congruences in the algebraic setting.

distinction between the category of regular subobjects of $\langle A, \alpha\rangle$ and their images under U_α, corresponding to the definition of bisimulation in Section 2.5.

We also will assume that $\mathcal{E}$ is regularly well-powered throughout.

REMARK 2.2.3. If $\mathcal{E}$ is a topos, then every mono is regular. So our definition of subcoalgebra coincides with the usual definition of subcoalgebra: Namely, $\langle B, \beta\rangle$ is a subcoalgebra of $\langle A, \alpha\rangle$ just in case there is a monic homomorphism

$$i : \langle B, \beta\rangle \rightarrowtail \langle A, \alpha\rangle .$$

In other words, if $\mathcal{E}$ is a topos, then

$$\mathsf{SubCoalg}(\langle A, \alpha\rangle) = \mathsf{Sub}(\langle A, \alpha\rangle).$$

Let $\mathsf{RegSub}(A)$ be the poset of regular subobjects of A. We define a functor

$$U_\alpha : \mathsf{SubCoalg}(\langle A, \alpha\rangle) \longrightarrow \mathsf{RegSub}(A),$$

taking a regular subcoalgebra

$$\langle B, \beta\rangle \xrightarrow{i} \langle A, \alpha\rangle$$

to the regular subobject

$$B \xrightarrow{i} A$$

(again, using the assumption that Γ preserves regular monos).

THEOREM 2.2.4. *The subcoalgebra forgetful functor U_α is full and injective on objects. In other words, $\mathsf{SubCoalg}(\langle A, \alpha\rangle)$ is a full subcategory of $\mathsf{RegSub}(A)$.*

PROOF. U_α is full by Corollary 1.2.10 (a map into a mono is a homomorphism when the composite is).

Let $U_\alpha(\langle B, \beta\rangle) = U_\alpha(\langle B, \beta'\rangle) = B$ and let

$$i : B \rightarrowtail A$$

be the regular mono homomorphic inclusion for B. Then, Γi is a regular mono (and hence a mono). Since

$$\Gamma i \circ \beta = \alpha \circ i = \Gamma i \circ \beta',$$

$\beta = \beta'$. $\square$

THEOREM 2.2.5. *Let $\mathcal{E}$ be cocomplete and almost co-regular and Γ preserve regular monos. The functor U_α creates joins and commutes with $\exists_f$.*

PROOF. The join of regular subcoalgebras $\langle P_i, \rho_i\rangle$ of $\langle A, \alpha\rangle$ is given as the epi-regular mono factorization of the map

$$\textstyle\coprod_i \langle P_i, \rho_i\rangle \longrightarrow \langle A, \alpha\rangle .$$

Because coproducts are created by U and U preserves regular epi-mono factorizations, U_α preserves the join $\bigvee_i P_i$.

Let $f:\langle A,\,\alpha\rangle \to \langle B,\,\beta\rangle$ and $i:\langle P,\,\rho\rangle \rightarrowtail \langle A,\,\alpha\rangle$ be given. Then

$$U_\beta \exists_f \langle P,\,\rho\rangle = U_\beta\,\mathsf{Im}(f\circ i) = \mathsf{Im}\,U(f\circ i) = \exists_{Uf} U_\alpha \langle P,\,\rho\rangle.$$

□

THEOREM 2.2.6. *If, in addition to the assumptions of Theorem 2.2.5,* Γ *preserves pullbacks of regular monos, then* U_α *also creates finite meets. Furthermore,* U_α *commutes with pullback of subobjects, i.e., for every* $f:\langle A,\,\alpha\rangle \to \langle B,\,\beta\rangle$,

$$U_\alpha \circ f^* = (Uf)^* \circ U_\beta.$$

PROOF. By Corollary 1.2.8, U creates pullbacks along regular monos. □

2.2.1. About the functor $[-]_\alpha$. In universal algebras, one can construct a least subcoalgebra containing a subset of the carrier of an algebra. This construction was discussed in Theorem 1.3.6, where we showed that the functor

$$\langle - \rangle_\alpha : \mathsf{Sub}(A) \longrightarrow \mathsf{SubAlg}(\langle A,\,\alpha\rangle)$$

was left adjoint to the forgetful functor

$$U_\alpha : \mathsf{SubAlg}(\langle A,\,\alpha\rangle) \longrightarrow \mathsf{Sub}(A)$$

.

The functor $\langle - \rangle$ was constructed under the assumption that $\mathsf{Sub}(A)$ had all meets. Using the fact that the meet of subalgebras again yields a subalgebra, $\langle P \rangle$ is defined as the meet of all subalgebras containing P.

In this section, we will carry out the analogous construction for regular subcoalgebras. Here, we use the fact that the join of regular subcoalgebras is again a regular subcoalgebra.

THEOREM 2.2.7. *Let* $\mathcal{E}$ *be cocomplete, regularly well-powered and have epi-regular mono factorizations and let* Γ *preserve regular monos. Then the forgetful functor*

$$U_\alpha : \mathsf{SubCoalg}(\langle A,\,\alpha\rangle) \longrightarrow \mathsf{RegSub}(A)$$

has a right adjoint,

$$[-]_\alpha : \mathsf{RegSub}(A) \longrightarrow \mathsf{SubCoalg}(\langle A,\,\alpha\rangle).$$

PROOF. The proof is a straightforward construction following the proof of Theorem 1.3.6, but we include it nonetheless.

Let $P \overset{i}{\rightarrowtail} \mathsf{RegSub}(A)$ be given. Define $[P]_\alpha$ to be the join of the collection

$$\mathfrak{P} = \{\,\langle B,\,\beta\rangle \rightarrowtail \langle A,\,\alpha\rangle \mid B \leq P\}.$$

PROOF.

$$
\begin{tikzcd}
\mathsf{SubCoalg}(\langle A,\,\alpha\rangle) \arrow[r, shift left, "\exists_f"] \arrow[r, phantom, "\perp"] \arrow[d, shift right, "U_\alpha"'] \arrow[d, phantom, "\dashv"] & \mathsf{SubCoalg}(\langle B,\,\beta\rangle) \arrow[l, shift left, "f^*"] \arrow[d, shift right, "U_\beta"'] \arrow[d, phantom, "\dashv"] \\
\mathsf{RegSub}(A) \arrow[r, shift left, "\exists_{Uf}"] \arrow[r, phantom, "\perp"] \arrow[u, shift right, "{[-]_\alpha}"'] & \mathsf{RegSub}(B) \arrow[l, shift left, "Uf^*"] \arrow[u, shift right, "{[-]_\beta}"']
\end{tikzcd}
$$

FIGURE 1. $[-]$ commutes with pullback

Then, if $\langle Q,\,\nu\rangle$ is any regular subcoalgebra of $\langle A,\,\alpha\rangle$ such that $Q \leq P$, then $\langle Q,\,\nu\rangle \in \mathfrak{P}$ and so $\langle Q,\,\nu\rangle \leq [P]_\alpha$. On the other hand, if $\langle Q,\,\nu\rangle \leq [P]_\alpha$, then

$$Q \leq U_\alpha[P]_\alpha \leq P.$$

□

For each of the three corollaries which follow, we work under the assumptions of Theorem 2.2.7.

COROLLARY 2.2.8. *For any homomorphism* $f\colon\langle A,\,\alpha\rangle \to \langle B,\,\beta\rangle$,

$$[-]_\beta \circ Uf^* = f^* \circ [-]_\alpha.$$

In Figure 1, the left adjoints commute by Theorem 2.2.5, and so the right adjoints commute as well. □

In Theorem 1.4.6, we showed that a homomorphism equalizes a relation R just in case it equalizes the least pre-congruence $[R]$ containing R. Our definition of subcoalgebra is dual to quotient of an algebra (which is, under certain assumptions, equivalent to congruences). Thus, it is theorems about congruences which yield theorems about subcoalgebras, rather than theorems about subalgebras[3].

COROLLARY 2.2.9. *Let* $\langle A,\,\alpha\rangle$ *be a* Γ*-coalgebra, with* P *a regular subobject of* A*. Let* $\langle B,\,\beta\rangle$ *be a* Γ*-coalgebra and* $f\colon\langle B,\,\beta\rangle \to \langle A,\,\alpha\rangle$ *a* Γ*-homomorphism. Then* Uf *factors through* P *iff* f *factors through* $[P]_\alpha$.

[3]Theorem 2.2.7 can be viewed as the dual of the theorem that we can construct least congruences containing a relation. In this sense, it is the dual of a theorem about congruences, rather than a theorem about subalgebras. One simply looks at the corresponding theorem regarding quotients of a congruence to see this.

PROOF.

$$\mathsf{Im}(Uf) = U_\alpha\,\mathsf{Im}(f) \le P \text{ iff } \mathsf{Im}\,f \le [P]_\alpha.$$

□

COROLLARY 2.2.10. *Let $\langle A,\, \alpha\rangle$ and $\langle B,\, \beta\rangle$ be given, with $f\colon\langle A,\, \alpha\rangle \to \langle B,\, \beta\rangle$ a homomorphism. Let $\langle D,\, \delta\rangle \le \langle A,\, \alpha\rangle$ and $P \le B$. Then*

$$\exists_{Uf} D \le P \text{ iff } \exists_f \langle D,\, \delta\rangle \le [P]_\beta.$$

PROOF. Follows immediately from Corollary 2.2.9. □

The next theorem gives some equivalent constructions of $[P]_\alpha$. The requirement that U be comonadic is only necessary for those constructions which explicitly use the right adjoint H — namely, for (3) and (4).

THEOREM 2.2.11. *Let $\mathcal{E}$ be regularly well-powered, cocomplete and have pullbacks and epi-regular mono factorizations. Let Γ be a covarietor that preserves regular monos with $U \dashv H$. Let*

$$\langle B,\, \beta\rangle \rightarrowtail^{b} \langle A,\, \alpha\rangle$$

and

$$P \rightarrowtail^{i} A$$

be given. The following are equivalent.

(1) $\langle B,\, \beta\rangle \cong [P]_\alpha$.

(2) Let $P \overset{i}{\rightarrowtail} A$ *be the equalizer of* $A \underset{c_2}{\overset{c_1}{\rightrightarrows}} C$ *and let* $\langle A,\, \alpha\rangle \underset{\widetilde{c_2}}{\overset{\widetilde{c_1}}{\rightrightarrows}} HC$ *be the adjoint transposes of c_1 and c_2, respectively. Then*

$$\langle B,\, \beta\rangle \overset{b}{\rightarrowtail} \langle A,\, \alpha\rangle \underset{\widetilde{c_2}}{\overset{\widetilde{c_1}}{\rightrightarrows}} HC$$

is an equalizer.

(3) There is a (necessarily regular mono) map $k\colon\langle B,\, \beta\rangle \rightarrowtail HP$ such that the following diagram is a pullback, where $\eta\colon 1 \Rightarrow HU$ is the unit of the adjunction $U \dashv H$.

$$\begin{tikzcd}
\langle B,\, \beta\rangle \arrow[r, tail, "k"] \arrow[d, tail, "b"'] \arrow[dr, phantom, "\lrcorner", very near start] & HP \arrow[d, tail, "Hi"] \\
\langle A,\, \alpha\rangle \arrow[r, tail, "\eta_\alpha"'] & HA
\end{tikzcd}$$

$$
\begin{tikzcd}
\langle D,\,\delta\rangle \arrow[drrr, bend left, "f"] \arrow[dr, dotted] \arrow[ddr, bend right, "g"'] & & & \\
& \langle B,\,\beta\rangle \arrow[r, tail, "\eta_\beta"] \arrow[d, tail, "b"'] & HB \arrow[r, tail] \arrow[dr, tail, "Hb"] & HP \arrow[d, tail] \\
& \langle A,\,\alpha\rangle \arrow[rr, tail, "\eta_\alpha"'] \arrow[d, shift right, "\widetilde{c_2}"'] \arrow[d, shift left, "\widetilde{c_1}"] & & HA \arrow[d, shift right, "Hc_2"'] \arrow[d, shift left, "Hc_1"] \\
& HC \arrow[rr, equal] & & HC
\end{tikzcd}
$$

FIGURE 2. The construction of $[P]$ as a pullback along the unit.

(4) (If $\mathcal{E}$ has a regular subobject classifier Ω) Let $\overline{\imath}$ be the classifying map for i, so the diagram below is a pullback.

$$
\begin{tikzcd}
P \arrow[r, "!"] \arrow[d, tail, "i"'] \arrow[dr, phantom, very near start, "\lrcorner"] & 1 \arrow[d, tail, "\mathsf{true}"] \\
A \arrow[r, "\overline{\imath}"'] & \Omega
\end{tikzcd}
$$

Then, the diagram below is also a pullback,

$$
\begin{tikzcd}
\langle B,\,\beta\rangle \arrow[r, "!"] \arrow[d, tail, "b"'] \arrow[dr, phantom, very near start, "\lrcorner"] & H1 \arrow[d, "H\mathsf{true}"] \\
\langle A,\,\alpha\rangle \arrow[r, "\widetilde{\imath}"'] & H\Omega
\end{tikzcd}
$$

where $\widetilde{\imath}$ is the adjoint transpose of $\overline{\imath}$.

PROOF. **(1)⇒(2):** Let $g\colon\langle D,\,\delta\rangle\to\langle A,\,\alpha\rangle$ be given, and suppose that g equalizes

$$\langle A,\,\alpha\rangle \underset{\widetilde{c_2}}{\overset{\widetilde{c_1}}{\rightrightarrows}} HC\ .$$

Then,

$$c_1\circ g = \varepsilon_C\circ\widetilde{c_1}\circ g = \varepsilon_C\circ\widetilde{c_2}\circ g = c_2\circ g$$

and so g factors through P. Hence, g factors uniquely through $[P]$ (Corollary 2.2.9).

(2)⇒(3): Let $\langle B,\,\beta\rangle$ be the equalizer of $\widetilde{c_1}$ and $\widetilde{c_2}$, as in (2). We claim that the top rectangle in the Figure 2 forms a pullback. Let $\langle D,\,\delta\rangle$, f and g be given so that Figure 2 commutes. Then, g equalizes $\widetilde{c_1}$ and $\widetilde{c_2}$ and so factors uniquely through b, as shown. It is easy to show that the upper triangle also commutes.

$$
\begin{tikzcd}
{[P]} \arrow[r, tail, "\eta_{[P]}"] \arrow[dr, dotted] \arrow[ddr, tail, bend right] & HU[P] \arrow[dr, tail] \arrow[ddr, tail, bend right] & \\
& \langle B,\,\beta\rangle \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, very near start, "\lrcorner"] & HP \arrow[d, tail, "Hi"] \\
& \langle A,\,\alpha\rangle \arrow[r, tail, "\eta_\alpha"'] & HA
\end{tikzcd}
$$

FIGURE 3. The construction of $[P]$ as a pullback along Hi.

(3)⇒(1): In Figure 3, the right hand triangle commutes because $U[P] \leq P$. The diagonal square commutes by naturality of the unit η. Hence, we have a unique map $[P] \rightarrowtail \langle B,\, \beta\rangle$, as shown, making the diagram commute. Thus, $[P] \leq \langle B,\, \beta\rangle$.

On the other hand, let $\widetilde{k}\colon B \to P$ be the adjoint transpose of

$$k\colon \langle B,\, \beta\rangle \longrightarrow HP.$$

Because

$$
\begin{aligned}
Hi \circ H\widetilde{k} \circ \eta_\beta &= Hi \circ k \\
&= \eta_\alpha \circ b \\
&= Hb \circ \eta_\beta,
\end{aligned}
$$

we see that $\widetilde{k} \circ i = b$. In other words, $B \leq P$. Hence, $\langle B,\, \beta\rangle \leq [P]$ and so $\langle B,\, \beta\rangle \cong [P]$.

(3)⇔(4): The right adjoint H preserves pullbacks. Consequently, the left hand square in Figure 4 is a pullback iff the whole rectangle is a pullback [**Bor94**, Proposition 2.5.9, Volume 1].

$$
\begin{tikzcd}
\langle B,\,\beta\rangle \arrow[r, tail] \arrow[d, tail, "b"'] \arrow[rr, bend left, "!"] & HP \arrow[r, "H!"'] \arrow[d, tail, "Hi"'] \arrow[dr, phantom, very near start, "\lrcorner"] & H1 \arrow[d, tail, "H\mathsf{true}"] \\
\langle A,\,\alpha\rangle \arrow[r, tail, "\eta_\alpha"] \arrow[rr, bend right, "\widetilde{\imath}"'] & HA \arrow[r, "H\overline{\imath}"] & H\Omega
\end{tikzcd}
$$

FIGURE 4. $[P]$ as a pullback along $H\mathsf{true}$.

∎

REMARK 2.2.12. In the proof of (3)⇒(1), above, we assumed the existence of $[P]$. This is not necessary. With a bit more work, one can loosen the assumptions of Theorem 2.2.11 (removing the assumption of coproducts) and replace (1) in with

(1)′ $[P]$ exists and $[P] \cong \langle B, \beta \rangle$.

REMARK 2.2.13. In Theorem 2.2.11, the construction of $[P]$ found in (3) is essentially the same construction one finds on [**BW85**, p. 216].

2.2.2. The associated modal operator. Let $\mathcal{E}$ be regularly well-powered, cocomplete and almost co-regular and let Γ preserve regular monos and pullbacks along regular monos. Let $\langle A, \alpha \rangle$ be a Γ-coalgebra. The adjunction $U_\alpha \dashv [-]_\alpha$ yields a comonad in the usual way. We will denote the functor part of this comonad,

$$U_\alpha[-]_\alpha : \mathsf{RegSub}(A) \longrightarrow \mathsf{RegSub}(A),$$

by $\Box_\alpha$ (sometimes dropping the subscript).

REMARK 2.2.14. The associated monad

$$[-]_\alpha U_\alpha : \mathsf{SubCoalg}(\langle A, \alpha \rangle) \longrightarrow \mathsf{SubCoalg}(\langle A, \alpha \rangle)$$

yields the trivial closure operator

$$1 : \mathsf{SubCoalg}(\langle A, \alpha \rangle) \longrightarrow \mathsf{SubCoalg}(\langle A, \alpha \rangle)$$

on subcoalgebras.

Because $\Box_\alpha$ is a functor on a poset, it is monotone. The counit and comultiplication transformations yield, for every P,

$$\Box_\alpha P \leq P$$
$$\Box_\alpha P \leq \Box_\alpha \Box_\alpha P$$

Furthermore, because U_α preserves finite meets, so does $\Box_\alpha = U_\alpha[-]_\alpha$. Hence, we have shown that $\Box$ is an **S4** modal necessity operator.

DEFINITION 2.2.15. An operator $\Box : P \to P$ on a Heyting algebra P is an **S4** *operator* if it satisfies the following:

(1) $\Box$ is monotone (i.e., is an endofunctor);
(2) $\Box$ is deflationary (i.e., $\Box \leq 1$);
(3) $\Box$ is idempotent (i.e., $\Box\Box = \Box$);
(4) $\Box(A \to B) \leq \Box A \to \Box B$;
(5) $\top \leq \Box\top$.

In other words, an **S4** operator is just a left exact comonad on a Heyting algebra.

THEOREM 2.2.16. $\Box_\alpha : \mathsf{RegSub}(A) \to \mathsf{RegSub}(A)$ *is an* **S4** *operator.*

Proof. (4) follows from the fact that $\Box$ preserves meets. The argument for (4) from this is standard, but we include it here.

By (1), we have

$$\Box((\varphi \to \psi) \wedge \varphi) \vdash \Box\psi,$$

and, hence,

$$\Box(\varphi \to \psi) \wedge \Box\varphi \vdash \Box\psi.$$

Therefore, $\Box(\varphi \to \psi) \vdash \Box\varphi \to \Box\psi$.

The top element $\top$ of $\mathsf{RegSub}(A)$ is just A itself. Clearly, $\Box_\alpha A = A$, and so (5) holds. $\Box$

Theorem 2.2.17. *Let $f : \langle A, \alpha\rangle \to \langle B, \beta\rangle$ be given. Then*

$$\Box_\alpha \circ (Uf)^* = (Uf)^* \circ \Box_\beta.$$

In other words, $\Box$ is a natural transformation

$$\Box : \mathsf{RegSub}(-) \circ U \Longrightarrow \mathsf{RegSub}(-) \circ U.$$

Proof. Both U_α and $[-]_\alpha$ preserve pullbacks along regular monos. See Figure 5. The front, right and rear faces are pullbacks and the bottom face commutes, so the left face is also a pullback. $\Box$

$$\begin{tikzcd}
\Box_\beta P \arrow[rr, tail] \arrow[dd, tail] & & UHP \arrow[dd, tail] & \\
& \Box_\alpha f^* P \arrow[ul] \arrow[rr, tail] \arrow[dd, tail] & & UHf^*P \arrow[ul] \arrow[dd, tail] \\
B \arrow[rr, tail] & & UHB & \\
& A \arrow[ul, "Uf"] \arrow[rr, tail] & & UHA \arrow[ul]
\end{tikzcd}$$

Figure 5. $\Box$ commutes with pullback.

Example 2.2.18. In Example 1.1.10 we discussed coalgebras for the set functor

$$\Gamma = \mathcal{P}(\mathbf{AtProp}) \times \mathcal{P} - .$$

Such coalgebras are Kripke models for the modal language $\mathcal{L}(\mathbf{AtProp})$. Given a Γ-coalgebra $\langle A, \langle \alpha_1, \alpha_2\rangle\rangle$, we consider the elements of A to be worlds. The first component,

$$\alpha_1 : A \longrightarrow \mathcal{P}(\mathbf{AtProp}),$$

of the structure map picks out those atomic formulas which are true in a world, while the second component,

$$\alpha_2 : A \longrightarrow \mathcal{P}(A),$$

gives the *accessibility relation*. A world b is accessible to a (written $a \to b$) just in case $b \in \alpha_2(a)$.

Let $\phi \in \mathcal{L}(\mathbf{AtProp})$ and $\mathfrak{A} = \langle A, \langle \alpha_1, \alpha_2 \rangle\rangle$ a Γ-coalgebra. Let $\mathsf{Mod}_{\mathfrak{A}}(\phi)$ be the collection

$$\{a \in A \mid a \models_{\mathfrak{A}} \phi\}.$$

We can characterize $\mathsf{Mod}_{\mathfrak{A}}(\phi)$ by induction on the structure of ϕ as follows.

- $\mathsf{Mod}_{\mathfrak{A}}(\top) = A$.
- $\mathsf{Mod}_{\mathfrak{A}}(\phi) = \alpha_1^{-1}(\phi)$ if $\phi \in \mathbf{AtProp}$.
- $\mathsf{Mod}_{\mathfrak{A}}(\neg\phi) = A \setminus \mathsf{Mod}_{\mathfrak{A}}(\phi)$.
- $\mathsf{Mod}_{\mathfrak{A}}(\Diamond\phi) = \{a \in A \mid \alpha_2(a) \cap \mathsf{Mod}_{\mathfrak{A}}(\phi) \neq \emptyset\}$.
- $\mathsf{Mod}_{\mathfrak{A}}(\bigwedge S) = \bigcap_{\phi \in S} \mathsf{Mod}_{\mathfrak{A}}(\phi)$.

Thus, for each $\phi \in \mathcal{L}(\mathbf{AtProp})$, we have $\mathsf{Mod}_{\mathfrak{A}}(\phi) \subseteq A$. We calculate $\Box_{\mathfrak{A}} \mathsf{Mod}_{\mathfrak{A}}(\phi)$, the (carrier of the) largest subcoalgebra of $\mathsf{Mod}_{\mathfrak{A}}(\phi)$. Note: this predicate over A should not be confused with the proposition $\Box\phi$, where $\Box$ is defined as $\neg\Diamond\neg\phi$ in $\mathcal{L}(\mathbf{AtProp})$. As we will show, despite the syntactic similarity,

$$\Box_{\mathfrak{A}} \mathsf{Mod}_{\mathfrak{A}}(\phi) \neq \mathsf{Mod}_{\mathfrak{A}}(\Box\phi),$$

although the two are related.

Let $\to^*$ be the reflexive and transitive closure of $\to$. We extend the language $\mathcal{L}(\mathbf{AtProp})$ by adding a new modal operator $\boxdot$. We extend the semantics to include this operator by adding the rule:

- $a \models_{\mathfrak{A}} \boxdot\phi$ iff, for all b such that $a \to^* b$, $b \models \phi$. In particular, $a \models_{\mathfrak{A}} \boxdot\phi$ implies $a \models_{\mathfrak{A}} \phi$.

The proposition $\boxdot\phi$ represents the condition that, not only is ϕ necessary, but ϕ is necessarily necessary and so on. Indeed, one can easily show

$$a \models_{\mathfrak{A}} \boxdot\phi \text{ iff } a \models_{\mathfrak{A}} \bigwedge_{i<\omega} \Box^i\phi.$$

If the accessibility relation for $\mathfrak{A}$ is reflexive and transitive, then $\boxdot\phi$ is equivalent to $\Box\phi$.

We claim that

$$\Box_{\mathfrak{A}} \mathsf{Mod}_{\mathfrak{A}}(\phi) = \mathsf{Mod}_{\mathfrak{A}}(\boxdot\phi).$$

First, suppose $a \in \mathsf{Mod}_{\mathfrak{A}}(\boxdot\phi)$ and $a \to b$. Then, clearly, $b \models_{\mathfrak{A}} \boxdot\phi$ as well, so

$$\alpha_2(a) \subseteq \mathsf{Mod}_{\mathfrak{A}}(\boxdot\phi).$$

In other words, $\mathsf{Mod}_{\mathfrak{A}}(\boxdot\phi)$ is (the carrier of) a subcoalgebra of $\langle A, \langle\alpha_1, \alpha_2\rangle\rangle$. So, since

$$\mathsf{Mod}_{\mathfrak{A}}(\boxdot\phi) \subseteq \mathsf{Mod}_{\mathfrak{A}}(\phi),$$

we have

$$\mathsf{Mod}_{\mathfrak{A}}(\boxdot\phi) \subseteq \Box_{\mathfrak{A}}\, \mathsf{Mod}_{\mathfrak{A}}(\phi).$$

To prove equality, one must show that $\mathsf{Mod}_{\phi}(\boxdot\phi)$ is the greatest subcoalgebra of $\langle A, \langle\alpha_1, \alpha_2\rangle\rangle$ contained in $\mathsf{Mod}_{\mathfrak{A}}(\phi)$.

Let $\langle B, \langle\beta_1, \beta_2\rangle\rangle$ be a subcoalgebra of $\langle A, \langle\alpha_1, \alpha_2\rangle\rangle$ such that $B \subseteq \mathsf{Mod}_{\mathfrak{A}}(\phi)$. To complete the proof, it suffices to show that $B \subseteq \mathsf{Mod}_{\mathfrak{A}}(\boxdot\phi)$. Let $b \in B$ and suppose that $b \rightarrow^* c$. Then $c \in B \subseteq \mathsf{Mod}_{\mathfrak{A}}(\phi)$, so $c \models_{\mathfrak{A}} \phi$. Hence, $b \models_{\mathfrak{A}} \boxdot\phi$ and so $b \in \mathsf{Mod}_{\mathfrak{A}}(\boxdot\phi)$, as desired.

EXAMPLE 2.2.19. Let $\langle A, \mathcal{O}_A\rangle$ be a topological space and $\langle A, \alpha\rangle$ the associated $\mathcal{F}$-coalgebra (see Examples 1.1.12 and 2.2.2). Then $U_\alpha[-]_\alpha$ is the interior operator. That is, if $S \subseteq A$, then $U_\alpha[S]_\alpha$ is the largest open subset of S.

2.2.3. The structure of $\mathsf{SubCoalg}(\langle A, \alpha\rangle)$. In this section, we will show that, if $\mathsf{RegSub}(A)$ is a complete Heyting algebra, then so is $\mathsf{SubCoalg}(\langle A, \alpha\rangle)$. This is an indication that subcoalgebras are the "right" objects to consider as unary predicates in the category $\mathcal{E}_\Gamma$. We extend this result to bisimulations in Section 2.5.

Throughout this section, we assume that $\mathcal{E}$ is regularly well-powered, almost coregular and cocomplete and that $\Gamma\colon\mathcal{E}\to\mathcal{E}$ preserves regular monos. Thus, by Theorem 2.2.7, the subcoalgebra forgetful functor

$$U_\alpha\colon\mathsf{SubCoalg}(\langle A, \alpha\rangle) \longrightarrow \mathsf{RegSub}(A)$$

has a right adjoint, $[-]_\alpha$. We further assume that Γ preserves pullbacks of regular monos.

DEFINITION 2.2.20. A *complete Heyting algebra* is a complete lattice $\langle S, \wedge, \bigvee\rangle$ which satisfies the infinitary distributive law

$$s \wedge \bigvee_{i\in I} t_i = \bigvee_{i\in I}(s \wedge t_i).$$

THEOREM 2.2.21. *If* $\mathsf{RegSub}(A)$ *is a complete Heyting algebra, then so is the category* $\mathsf{SubCoalg}(\langle A, \alpha\rangle)$.

PROOF. The subcoalgebra forgetful functor U_α creates joins and finite meets, so $\mathsf{SubCoalg}(\langle A, \alpha\rangle)$ inherits the infinitary distributive law from $\mathsf{RegSub}(A)$. $\Box$

DEFINITION 2.2.22. A *Heyting algebra* is a lattice with $\top$ and $\bot$ such that $\wedge$ has a right adjoint $\rightarrow$.

REMARK 2.2.23. Definition 2.2.20 is equivalent to the statement that S is a complete lattice which is a Heyting algebra.

THEOREM 2.2.24. *If* $\mathsf{RegSub}(A)$ *is a Heyting algebra, then so is* $\mathsf{SubCoalg}(\langle A, \alpha\rangle)$.

PROOF. We need to show that $\wedge$ in $\mathsf{SubCoalg}(\langle A, \alpha\rangle)$ has a right adjoint. Let $\langle B, \beta\rangle$ and $\langle C, \gamma\rangle$ be subcoalgebras of $\langle A, \alpha\rangle$. We calculate

$$\begin{aligned}
\langle B, \beta\rangle \wedge \langle C, \gamma\rangle \leq \langle D, \delta\rangle &\text{ iff } B \wedge C \leq D && \text{since } U_\alpha \text{ creates meets,}\\
&\text{ iff } B \leq C \to D && \text{since } - \wedge C \dashv C \to -,\\
&\text{ iff } \langle B, \beta\rangle \leq [C \to D]_\alpha && \text{since } U_\alpha \dashv [-]_\alpha.
\end{aligned}$$

□

REMARK 2.2.25. Theorem 2.2.24 implies that the negation for $\mathsf{SubCoalg}(\langle A, \alpha\rangle)$ is given as

$$\neg\langle B, \beta\rangle = [\neg B]_\alpha.$$

EXAMPLE 2.2.26. The category $\mathsf{SubCoalg}(\langle A, \alpha\rangle)$ is not usually boolean, even if $\mathsf{RegSub}(A)$ is boolean. Consider the functor $\Gamma A = \mathbb{N} \times A$ and the coalgebra $\langle A, \alpha\rangle$ where $A = \{a, b\}$ and

$$\begin{aligned}
\alpha(a) &= \langle 17, b\rangle,\\
\alpha(b) &= \langle 17, b\rangle.
\end{aligned}$$

Let $\langle B, \beta\rangle$ be the subcoalgebra $B = \{b\}$ and $\beta(b) = \alpha(b)$. Then

$$\neg\langle B, \beta\rangle = [\{a\}]_\alpha = \langle 0, !\rangle,$$

so $\langle B, \beta\rangle \vee \neg\langle B, \beta\rangle \neq \langle A, \alpha\rangle$.

2.3. Subcoalgebras generated by a subobject

Let $\langle A, \alpha\rangle$ be a Γ-algebra and $P \subseteq A$. If $\mathsf{RegSub}(A)$ is a complete lattice, then one can construct $\langle P\rangle$, the least subalgebra of $\langle A, \alpha\rangle$ containing P (see Theorem 1.3.6). This construction yields a left adjoint to the forgetful functor for subalgebras:

$$\begin{tikzcd}
\mathsf{Sub}(A) \arrow[r, bend left, "\langle - \rangle"] \arrow[r, phantom, "\perp"] & \mathsf{SubAlg}(\langle A, \alpha\rangle) \arrow[l, bend left, "U"]
\end{tikzcd}.$$

As we've shown, the coalgebraic analogue for $\langle - \rangle$ is $[-]$, a right adjoint to the subcoalgebra forgetful functor. Whereas, in categories of algebras, a closure operation naturally arises (by closing a subobject under the algebraic operations), in categories of coalgebras, an interior operation is the "natural" operation.

Nonetheless, for certain functors $\Gamma : \mathcal{E} \to \mathcal{E}$, there is a left adjoint

$$\langle - \rangle_\alpha : \mathsf{Sub}(A) \longrightarrow \mathsf{SubCoalg}(\langle A, \alpha\rangle)$$

to the forgetful functor, taking a subobject to the least subcoalgebra containing it. We describe the operation here.

The following theorem is almost an immediate corollary of Theorem 1.2.7 (U creates whatever limits Γ preserves). The weakening of the assumption that Γ preserves intersections to just Γ preserves *non-empty* intersections requires a bit of work to ensure that it goes through, but as one can see, the work is really just the proof of Theorem 1.2.7 again.

The use of non-empty intersections for categories of coalgebras first appears in the work of Gumm and Schröder, as seen in [**Gum01b**].

THEOREM 2.3.1. *If Γ preserves regular monos non-empty κ-intersections, then*

$$U : \mathcal{E}_\Gamma \longrightarrow \mathcal{E}$$

creates κ-intersections of regular subcoalgebras.

PROOF. Let $\{\langle C_i, \gamma_i\rangle\}_{i<\kappa}$ be a family of regular subcoalgebras of $\langle B, \beta\rangle$. If $\bigcap C_i = 0$, then clearly

$$\bigcap \langle C_i, \gamma_i\rangle = \langle 0, !\rangle.$$

Otherwise, let C be the intersection of the C_i's, with inclusions

$$c_i : C \rightarrowtail C_i .$$

Then, ΓC is the limit of the ΓC_i's, with the Γc_i's forming a limiting cone. Since the maps

$$\gamma_i \circ c_i : C \longrightarrow \Gamma C_i$$

form a cone for C over the ΓC_i's, there is a unique structure map $\gamma \xrightarrow{C} \Gamma C$ such that each c_i is a homomorphism.

It is routine to check that, for any regular subcoalgebra $\langle A, \alpha\rangle$ of $\langle B, \beta\rangle$ contained in each of the $\langle C_i, \gamma_i\rangle$'s, the inclusion

$$A \leq \bigcap C_i$$

is a homomorphism. For this, we use the fact that Γ preserves regular monos. □

EXAMPLE 2.3.2. The filter functor $\mathcal{F}$ doesn't preserve non-empty intersections. Indeed, from Example 1.1.12, we learn that the category of topological spaces and open, continuous maps is a subcategory of $\mathbf{Set}_{\mathcal{F}}$. The open subsets of a space form the subcoalgebras when we view the space as a filter coalgebra, but open sets are typically not closed under intersection.

THEOREM 2.3.3. *Let $\mathcal{E}$ be almost co-regular, regularly well-powered and have coproducts and let Γ preserve regular monos. Let $\langle A, \alpha\rangle \in \mathcal{E}_\Gamma$. The following are equivalent.*

(1) $U_\alpha : \mathsf{SubCoalg}(\langle A,\, \alpha\rangle) \to \mathsf{RegSub}(A)$ *creates intersections.*
(2) U_α *has a left adjoint,* $\langle - \rangle_\alpha$*.*
(3) (Assuming $\mathcal{E}$ *is well-pointed) For each global element* $a \in A$*, there is a least subcoalgebra containing* a *(denoted* $\langle a\rangle_\alpha$*).*

Proof. We prove that (1) and (2) are equivalent. Clearly, (2) implies (3). We complete the proof by assuming that $\mathcal{E}$ is well-pointed and show that (3) implies (2).

(1) ⇒(2): Let $P \le A$ and define $\langle P\rangle_\alpha$ to be the meet

$$\bigwedge_{P \le B} \langle B,\, \beta\rangle$$

in $\mathsf{SubCoalg}(\langle A,\, \alpha\rangle)$. The proof that $\langle - \rangle_\alpha \dashv U_\alpha$ is essentially the same as that in Theorem 1.3.6.

(2)⇒(1): Let $\{\langle B_i,\, \beta_i\rangle\}_{i\in I} \subseteq \mathsf{SubCoalg}(\langle A,\, \alpha\rangle)$. We will show that

$$\bigwedge_{i\in I} \langle B_i,\, \beta_i\rangle = \langle \bigwedge_{i\in I} B_i\rangle_\alpha, \tag{7}$$

$$U_\alpha \langle \bigwedge_{i\in I} B_i\rangle_\alpha = \bigwedge_{i\in I} B_i. \tag{8}$$

Since $\bigwedge_{i\in I} B_i \le B_i$, we have

$$\langle \bigwedge_{i\in I} B_i\rangle_\alpha \le \langle B_i\rangle_\alpha = \langle B_i,\, \beta_i\rangle.$$

Now, let $\langle C,\, \gamma\rangle \le \langle B_i,\, \beta_i\rangle$ for all $i \in I$. Then $C \le \bigwedge_{i\in I} B_i$. Hence,

$$\langle C,\, \gamma\rangle = \langle C\rangle_\alpha \le \langle \bigwedge_{i\in I} B_i\rangle_\alpha,$$

and so (7) holds.

For (8), we use the unit of the adjunction $\langle - \rangle_\alpha \dashv U_\alpha$ to conclude

$$\bigwedge_{i\in I} B_i \le U_\alpha \langle \bigwedge_{i\in I} B_i\rangle_\alpha.$$

Since $\langle \bigwedge_{i\in I} B_i\rangle_\alpha \le \langle B_i,\, \beta_i\rangle$ for all i, we have $U_\alpha\langle \bigwedge_{i\in I} B_i\rangle_\alpha \le B_i$ for all i. Hence, $U_\alpha\langle \bigwedge_{i\in I} B_i\rangle_\alpha \le \bigwedge_{i\in I} B_i$.

(3)⇒(2): Let $P \le A$. We define $\langle P\rangle_\alpha = \bigvee_{a\in P} \langle a\rangle_\alpha$ (where each a is a global element of P). Because $\mathcal{E}$ is well-pointed,

$$P = \bigvee_{a\in P} a \le \bigvee_{a\in P} U_\alpha\langle a\rangle_\alpha = U_\alpha \bigvee_{a\in P} \langle a\rangle_\alpha.$$

Hence, if $\langle P\rangle_\alpha \le \langle C,\, \gamma\rangle$, then $P \le U_\alpha\langle P\rangle_\alpha \le C$.

Let $\langle C, \gamma\rangle \leq \langle A, \alpha\rangle$ and $P \leq C$. We must show that $\langle P\rangle_\alpha \leq \langle C, \gamma\rangle$. For each global element $a \in P$, also $a \in C$. Thus, for each a, $\langle a\rangle_\alpha \leq \langle C, \gamma\rangle$ and so $\bigvee_{a\in P} \langle a\rangle_\alpha \leq \langle C, \gamma\rangle$.

$\square$

THEOREM 2.3.4. *Let $\langle -\rangle_\alpha \dashv U_\alpha$. The composite*

$$\langle -\rangle_\alpha \circ U_\alpha$$

is the identity $\mathsf{SubCoalg}(\langle A, \alpha\rangle) \twoheadrightarrow \mathsf{SubCoalg}(\langle A, \alpha\rangle)$.

PROOF. By the adjunction $\langle -\rangle_\alpha \dashv U_\alpha$, we have $\langle -\rangle_\alpha \circ U_\alpha \leq 1$. Also by the adjunction, we have

$$U_\alpha \leq U_\alpha \circ \langle -\rangle_\alpha \circ U_\alpha$$

and U_α is full (Theorem 2.2.4). $\square$

On the other hand, $U\langle\rangle$ is a non-trivial closure operator, which we denote $\lhd$, taking a subobject $A \leq U\langle B, \beta\rangle$ to its closure under the structure map β. We see that we have another adjunction, $\lhd\dashv \Box$. This closure operator is also discussed in [**Gum01b, Jac99**].

EXAMPLE 2.3.5. Let Γ:**Set**$\twoheadrightarrow$**Set** be the functor $A \mapsto Z \times A$ (see Example 1.1.7). Let $\langle A, \alpha\rangle$ be a Γ-coalgebra and $a \in A$. Then it is easy to see that

$$U_\alpha\langle a\rangle_\alpha = \{t^i_\alpha \mid i < \omega\}.$$

In other words, we close $\{a\}$ under the tail operation, t.

More generally, if $\mathbb{P}$ is any polynomial functor,

$$\mathbb{P}(A) = \coprod_{i<\omega} Z_i \times A^i,$$

we can define $\langle a\rangle_\alpha$ to be the collection of all $b \in A$ such that there is a path from a to b via the structure map α. To make this precise, define a relation $\to$ on A by

$$b {\to} c \text{ iff } \alpha(b) \in Z_i \times A^i \text{ and } \exists j < i(\pi_j \circ \alpha(b) = c).$$

Let $\to^*$ be the reflexive and transitive closure of $\to$. We claim that

$$U_\alpha\langle a\rangle_\alpha = \{b \mid a \to^* b\}.$$

We show that $\langle a\rangle_\alpha$ (by this definition) is a subcoalgebra of $\langle A, \alpha\rangle$. Let $a \to^* b$ and

$$\alpha(b) = \langle z, \langle b_1, \ldots, b_{i-1}\rangle\rangle.$$

Then $a \to^* b$ for each b_j $(j < i)$, so $\langle z, \langle b_1, \ldots, b_{i-1}\rangle\rangle \in \Gamma\langle a\rangle_\alpha$. In other words, $\langle a\rangle_\alpha$ is closed under the structure map α.

It is easy to check that $\langle a\rangle_\alpha \leq \langle C, \gamma\rangle$ iff $a \in C$ for all $\langle C, \gamma\rangle \leq \langle A, \alpha\rangle$.

EXAMPLE 2.3.6. Let $\Gamma = \mathcal{P}(\mathbf{AtProp}) \times \mathcal{P}-$ from Examples 1.1.10 and 2.2.18. In Example 2.2.18, we defined $\to^*$ as the transitive and reflexive closure of the accessibility relation, $\to$. It is easy to see that, for any Γ-coalgebra $\mathfrak{A} = \langle A, \langle\rangle, \alpha_1\rangle\alpha_2$,

$$U_{\mathfrak{A}}\langle a\rangle_{\mathfrak{A}} = \{b \mid a \to^* b\}.$$

This operation doesn't yield a "natural" operation on $\mathsf{Mod}_{\mathfrak{A}}(\phi)$ like $\Box_{\mathfrak{A}}$ did. One calculates

$$U_{\mathfrak{A}}\langle \mathsf{Mod}_{\mathfrak{A}}(\phi)\rangle_{\mathfrak{A}} = \{b \mid \exists a \,.\, a \models \phi \text{ and } a \to^* b\},$$

which seems a less interesting collection — one which is not expressible in terms of the modal operations of the language $\mathcal{L}(\mathbf{AtProp})$.

One has the impression that $\langle - \rangle_\alpha$ is often definable as a closure of a relation $\to$ like those found in Examples 2.3.5 and 2.3.6. It is difficult to make this intuition precise, since it involves defining an accessibility relation for a class of functors. In Example 2.3.5, we use the inductive definition of polynomial functors for the definition of $\to$. We can extend this class to include functors which are built from $\mathcal{P}$ in addition to constant and identity functors by $+$ and $\times$, as in Example 2.3.6. It is unclear how to do this for a class of functors generally[4] — the inductive construction of the functor seems to play a key role in the definition of $\to$.

2.4. Limits in categories of coalgebras revisited

The presence of a right adjoint to the coalgebraic forgetful functor allows one to construct limits in the category of coalgebras, $\mathcal{E}_\Gamma$, given that the corresponding limits exist in $\mathcal{E}$. We present here essentially a generalization of the proof that $\mathbf{Set}_\Gamma$ is complete if Γ is a covarietor, as found in [**GS01**].

While developing limits in categories of coalgebras, we also sketch the corresponding proofs that categories of algebras have colimits. However, we sometimes prefer to strengthen the assumptions on the algebraic theorems, so that we may reason about congruences (rather than a closure operator on quotients). This preference comes from a desire to explicitly see how reasoning about $\mathcal{E}^\Gamma$ comes directly from proofs about universal algebras, and categories of universal algebras do satisfy these stronger assumptions. In any case, we make clear that the theorem holds under the weaker assumptions as well, and also present the basic concepts necessary to prove it there.

[4]For similar reasons, Bart Jacobs focuses on inductively specified classes of functors in [**Jac99**] and elsewhere.

2.4.1. Equalizers in $\mathcal{E}_\Gamma$, coequalizers in $\mathcal{E}^\Gamma$. Equalizers of coalgebras was first discussed in [**Wor98**], where Worrell proves that equalizers exist when one-generated subcoalgebras exist and Γ is bounded (see Definition 3.7.20). The theorem below is a generalization of [**GS01**, Theorem 5.1], where it is proved for coalgebras over **Set**. A general proof of the completeness of $\mathcal{E}_\Gamma$, given that $\mathcal{E}$ is complete and certain other assumptions, can be found in [**JPT$^+$98**] as well as [**GS01**].

THEOREM 2.4.1. *Let $\mathcal{E}$ be regularly well-powered, cocomplete, have equalizers and epi-regular mono factorizations and let Γ preserve regular monos. Then $\mathcal{E}_\Gamma$ has all equalizers.*

PROOF. Let

$$\langle A,\, \alpha\rangle \overset{f}{\underset{g}{\rightrightarrows}} \langle B,\, \beta\rangle$$

be given and take the equalizer $P \rightarrowtail A$ of Uf and Ug in $\mathcal{E}$. Then, $[P]_\alpha$ is the equalizer of f and g in $\mathcal{E}_\Gamma$. Indeed, if h is a homomorphism that equalizes f and g, then Uh factors through P. From Corollary 2.2.9, we conclude that h factors through $[P]_\alpha$. Uniqueness easily follows. $\square$

THEOREM 2.4.2. *Let $\mathcal{E}$ be regularly co-well-powered, complete and have all coequalizers and regular epi-mono factorizations and let Γ preserve regular epis. Then, $\mathcal{E}^\Gamma$ has all coequalizers.*

PROOF. We sketch the proof. Let $\mathsf{Quot}(B)$ denote the category of quotients of B, i.e., $\mathsf{Quot}(B)$ consists of equivalence classes of regular epis. Let $\mathsf{Quot}(\langle B,\, \beta\rangle)$ be the corresponding category of quotients in $\mathcal{E}^\Gamma$. Show that there is a functor

$$\Theta : \mathsf{Quot}(B) \longrightarrow \mathsf{Quot}(\langle B,\, \beta\rangle)$$

left adjoint to the evident inclusion $\mathsf{Quot}(\langle B,\, \beta\rangle) \rightarrowtail \mathsf{Quot}(B)$. Specifically, given $B \twoheadrightarrow Q$ be given. Define ΘQ to be the regular epi-mono factorization of the evident map $\langle B,\, \beta\rangle \to \langle Q',\, \nu\rangle$, where $\langle Q',\, \nu\rangle$ is the limit of

$$\{\, \langle B,\, \beta\rangle \longrightarrow \langle P,\, \rho\rangle \mid P \in \mathsf{Quot}(Q)\}.$$

This is the formal dual of the construction of $[-]$, of course.

Show that, if Q is the coequalizer of $U\big(\, \langle A,\, \alpha\rangle \rightrightarrows \langle B,\, \beta\rangle\,\big)$, then ΘQ is the coequalizer of $\langle A,\, \alpha\rangle \rightrightarrows \langle B,\, \beta\rangle$.

An equivalent proof in a more restrictive setting may seem more familiar. Suppose, in addition to our other assumptions, that $\mathcal{E}$ is exact and that Γ preserves exact sequences. Given

$$\langle A,\, \alpha\rangle \overset{f}{\underset{g}{\rightrightarrows}} \langle B,\, \beta\rangle\ ,$$

take the kernel pair K of the coequalizer of f and g in $\mathcal{E}$, and then take the least congruence $\overline{K}$ containing K, according to Theorem 1.4.13. Since $\mathcal{E}^\Gamma$ is also exact (by Theorem 1.4.11), we can take the coequalizer $\langle Q,\, \nu\rangle$ of $\overline{K}$. It is little work to show that $\langle Q,\, \nu\rangle$is also the coequalizer of f and g. ☐

2.4.2. Products in $\mathcal{E}_\Gamma$, coproducts in $\mathcal{E}^\Gamma$. We find Theorems 2.4.3 and 2.4.6 in [**GS01**], where they are proved for coalgebras over **Set**. We extend the theorems to categories $\mathcal{E}$ which have a suitable structure inherited by $\mathcal{E}_\Gamma$, for appropriate functors Γ.

THEOREM 2.4.3. *Let $\mathcal{E}$ be cocomplete, κ-complete, regularly well-powered and have epi-regular mono factorizations and let Γ preserve regular monos. Let $\{\langle A_i,\, \alpha_i\rangle\}_{i<\kappa}$ be Γ-coalgebras and assume that $\prod\langle A_i,\, \alpha_i\rangle$ exists. We'll denote this product $\langle D,\, \delta\rangle$ with projections*

$$d_i : \langle D,\, \delta\rangle \longrightarrow \langle A_i,\, \alpha_i\rangle\,.$$

Let

$$\{c_i : \langle C_i,\, \gamma_i\rangle \rightarrowtail \langle A_i,\, \alpha_i\rangle\}_{i<\kappa}$$

be a family of regular subcoalgebras of the $\langle A_i,\, \alpha_i\rangle$'s. Then the product $\prod \langle C_i,\, \gamma_i\rangle$ exists in $\mathcal{E}_\Gamma$.

PROOF. Let P be the pullback (in $\mathcal{E}$) shown below.

$$\begin{tikzcd} P \arrow[r, tail, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, "\langle d_i\rangle"] \\ \prod C_i \arrow[r, tail, "\prod c_i"'] & \prod A_i \end{tikzcd}$$

We will show that $[P]_\delta$, the largest regular subcoalgebra of $\langle D,\, \delta\rangle$ contained in P, is the product of the $\langle C_i,\, \gamma_i\rangle$'s. We claim that the projections

$$r_i : [P] \longrightarrow \langle C_i,\, \gamma_i\rangle$$

are given by the composite

$$[P] \overset{j}{\rightarrowtail} P \xrightarrow{p_1} \prod C_i \xrightarrow{\pi_i} C_i,$$

but we must first establish that this composite is a coalgebra homomorphism. For this, we refer to Figure 6. We want to show that the front face of this diagram commutes. We use the fact that $\Gamma c_i : \Gamma C_i \to \Gamma A_i$ is a (regular) mono and show that

$$\Gamma c_i \circ \Gamma(\pi_i \circ p_1 \circ j \circ \rho) = \Gamma c_1 \circ \gamma_i \circ \pi_i \circ p_1 \circ j,$$

where ρ is the structure map for the coalgebra $[P]$. The squares on each end and the rectangle in back commute because the maps along the bottom (c_i, $p_2 \circ j$ and

$$
\begin{tikzcd}
 & & \Gamma D \arrow[rr] & & \Gamma\prod A_i \arrow[rr] & & \Gamma A_i \\
\Gamma U[P] \arrow[r, tail] \arrow[urr, tail] & \Gamma P \arrow[ur, tail] \arrow[rr] & & \Gamma\prod C_i \arrow[ur, tail] \arrow[rr] & & \Gamma C_i \arrow[ur, tail] & \\
 & & D \arrow[uu] \arrow[rr] & & \prod A_i \arrow[rr] & & A_i \arrow[uu] \\
U[P] \arrow[uu] \arrow[r, tail] \arrow[urr, tail] & P \arrow[ur, tail] \arrow[rr] & & \prod C_i \arrow[ur, tail] \arrow[rr] & & C_i \arrow[ur, tail] \arrow[uu] &
\end{tikzcd}
$$

FIGURE 6. The projection $r_i : U[P] \to C_i$ is a Γ-homomorphism.

$\pi_i \circ \langle d_i \rangle = d_i$, respectively) are coalgebra homomorphisms. The right hand square on the bottom face commutes by naturality, while the left hand square is a pullback.

To show that $[P]$ is a product, let $\langle B, \beta\rangle$ be a Γ-coalgebra and let a family of homomorphisms $\{f_i : \langle B, \beta\rangle \to \langle C_i, \gamma_i\rangle\}_{i<\kappa}$ be given. Then, by the definition of P, there is a unique map $B \to P$ so that the diagram below commutes.

$$
\begin{tikzcd}
B \arrow[dr, dotted] \arrow[drr, bend left, "U\langle c_i \circ f_i\rangle"] \arrow[ddr, bend right, "\langle f_i\rangle"'] & & \\
 & P \arrow[r, tail, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, "\langle d_i\rangle"] \\
 & \prod C_i \arrow[r, tail, "\prod c_i"'] & \prod A_i
\end{tikzcd}
$$

By Corollary 2.2.9, we get a factorization of $B \to P$ through $[P]$. Uniqueness easily follows. $\square$

For many functors of interest, the step of applying $[-]$ to P is unnecessary. The following theorem shows that, if Γ preserves non-empty intersections, then $[P] = P$. In particular, for finite products, if Γ preserves weak pullbacks, then the carrier for the product $\prod\langle C_i, \gamma_i\rangle$ is just P.

COROLLARY 2.4.4. *Let $\mathcal{E}$, Γ, $\{\langle A_i, \alpha_i\rangle\}$, etc., be given as in the statement of Theorem 2.4.3 and suppose, further, that Γ preserves pullbacks along regular monos*

$$\begin{tikzcd}
P \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & P_2 \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & C_2 \arrow[d, tail] \\
P_1 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[r] \arrow[d] & A_2 \\
C_1 \arrow[r, tail] & A_1 &
\end{tikzcd}$$

FIGURE 7. P is an intersection of the P_i's.

and non-empty κ-intersections. Then, the pullback

$$\begin{tikzcd}
P \arrow[r, tail, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, "\langle d_i \rangle"] \\
\prod C_i \arrow[r, tail, "\prod c_i"'] & \prod A_i
\end{tikzcd}$$

is invariant under $[-]$ ($[P] = P$). In fact, there is a (necessarily unique) structure map

$$\rho : P \longrightarrow \Gamma P$$

such that

$$\langle P,\, \rho \rangle = \prod \langle C_i,\, \gamma_i \rangle.$$

PROOF. For each i, let P_i be the pullback shown below.

$$\begin{tikzcd}
P_i \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, "d_i"] \\
C_i \arrow[r, tail, "c_i"'] & A_i
\end{tikzcd}$$

Because Γ preserves pullbacks along regular monos, U creates such pullbacks. Hence, each P_i is invariant. One can show that P is the intersection of the P_i's (see Figure 7 for an illustration of the case $\kappa = 2$). Theorem 2.3.1 completes the proof. $\square$

The following theorem dualizes the result of Theorem 2.4.3.

THEOREM 2.4.5. *Let $\mathcal{E}$ be complete, κ-cocomplete, regularly co-well-powered and have regular epi-mono factorizations and let Γ preserve regular epis. Let $\{\langle A_i,\, \alpha_i \rangle\}_{i<\kappa}$ be Γ-algebras and assume that $\coprod \langle A_i,\, \alpha_i \rangle$ exists. Let*

$$\{\ \langle A_i,\, \alpha_i \rangle \longrightarrow\!\!\!\!\triangleright \langle C_i,\, \gamma_i \rangle\ \}_{i<\kappa}$$

be a family of quotients of the $\langle A_i,\, \alpha_i \rangle$'s. Then the coproduct $\coprod \langle C_i,\, \gamma_i \rangle$ exists in $\mathcal{E}_\Gamma$.

PROOF. The proof of the theorem as stated is just the dualization of Theorem 2.4.3, using the functor Θ defined in the proof that $\mathcal{E}^\Gamma$ has coequalizers (Theorem 2.4.2). Instead of explicitly dualizing the theorem, we prefer to sketch the proof using congruences in the case that $\mathcal{E}$ is exact and Γ preserves exact sequences, so that $\mathcal{E}^\Gamma$ is also exact (Theorem 1.4.11). We also restrict our interest to the case $\kappa = 2$, just to simplify notation.

By assumption, we have a pair of regular epis

$$p : \langle A, \alpha\rangle \longrightarrow \langle C, \gamma\rangle,$$
$$q : \langle B, \beta\rangle \longrightarrow \langle D, \delta\rangle$$

and the coproduct $\langle A, \alpha\rangle + \langle B, \beta\rangle$ exists in $\mathcal{E}^\Gamma$. Let K be the kernel pair of $p + q$, shown below.

$$K \rightrightarrows A + B \xrightarrow{p+q} C + D$$

We would like to take the smallest congruence containing K, but K is not necessarily a relation on $U(\langle A, \alpha\rangle + \langle B, \beta\rangle)$. So, we first take the coequalizer

$$U(\langle A, \alpha\rangle + \langle B, \beta\rangle) \xrightarrow{r} R$$

of the diagram below.

$$K \rightrightarrows A + B \longrightarrow U(\langle A, \alpha\rangle + \langle B, \beta\rangle)$$

Then, we take the kernel pair of r,

$$L \rightrightarrows U(\langle A, \alpha\rangle + \langle B, \beta\rangle).$$

We claim that $\langle C, \gamma\rangle + \langle D, \delta\rangle$ is the coequalizer of the least congruence containing L, but we omit the proof. □

THEOREM 2.4.6. *Let $\mathcal{E}$ be cocomplete, κ-complete, regularly well-powered and have epi-regular mono factorizations. Let Γ preserve regular monos. Suppose further that $U : \mathcal{E}_\Gamma \to \mathcal{E}$ has a right adjoint H (i.e., Γ is a covarietor). Then $\mathcal{E}_\Gamma$ has κ-products.*

PROOF. Let H be the right adjoint to U and let $\{\langle A_i, \alpha_i\rangle\}_{i<\kappa}$ be a κ-family of coalgebras. Then, from Corollary 2.1.16, each $\langle A_i, \alpha_i\rangle$ is a regular subalgebra of the HA_i. Because H is a right adjoint, it preserves limits and so

$$H(\prod A_i) \cong \prod HA_i.$$

Hence, $\prod HA_i$ exists in $\mathcal{E}_\Gamma$ and we can apply Theorem 2.4.3. □

COROLLARY 2.4.7. *If $\mathcal{E}$ is cocomplete, κ-complete, regularly well-powered and has epi-regular mono factorizations, Γ a covarietor that preserves regular monos, then $\mathcal{E}_\Gamma$ is κ-complete.*

Theorem 2.4.6 shows that, given a right adjoint to the coalgebraic forgetful functor (and the conditions of Theorem 2.4.3), the category $\mathcal{E}_\Gamma$ has products. The algebraic analogue to Theorem 2.4.6 states that, if free algebras are available, then $\mathcal{E}^\Gamma$ has coproducts. This fact is well-known in the study of universal algebras. We state the theorem in the same generality as Theorem 2.4.6.

THEOREM 2.4.8. *Let $\mathcal{E}$ be complete, κ-cocomplete, regularly co-well-powered and have regular epi-mono factorizations and let Γ a varietor that preserves regular epis. Then $\mathcal{E}^\Gamma$ is κ-cocomplete.*

PROOF. Essentially the same as Theorem 2.4.6. We have coproducts of free algebras, and each algebra is the quotient of a free algebra. $\square$

The next theorem shows some equivalent constructions of the product of coalgebras.

THEOREM 2.4.9. *Let $\mathcal{E}$, Γ and H be given as in Theorem 2.4.6 and let*

$$\{\langle A_i,\, \alpha_i\rangle\}_{i<\kappa}$$

be a κ-family of coalgebras. Then the following are equivalent.

(1) $\langle B,\, \beta\rangle \cong \prod\langle A_i,\, \alpha_i\rangle$

(2) $\langle B,\, \beta\rangle = [P]_{\prod HA_i}$, where P is the pullback shown below.

(9)
$$\begin{tikzcd} P \arrow[r, tail, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & UH(\prod A_i) \arrow[d, "\langle UH\pi_i\rangle"] \\ \prod A_i \arrow[r, tail, "\prod U\eta_{\alpha_i}"'] & \prod UHA_i \end{tikzcd}$$

(3) $\langle B,\, \beta\rangle$ is the largest regular subcoalgebra of $H\prod A_i$ such that, for every $i \in I$,

$$B \rightarrowtail UH\prod A_i \xrightarrow{\varepsilon_{\prod A_i}} \prod A_i \xrightarrow{\pi_i} A_i$$

is a Γ-homomorphism.

(4) $\langle B,\, \beta\rangle$ fits into a pullback as shown below.

$$\begin{tikzcd} \langle B,\, \beta\rangle \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod HA_i \arrow[d, tail, "\prod HU\eta_{\alpha_i}"] \\ \prod HA_i \arrow[r, "\prod \eta_{HA_i}"'] & \prod HUHA_i \end{tikzcd}$$

(5) $\langle B,\, \beta\rangle = [E]_{\prod HA_i}$, where E is the equalizer of the diagram below.

$$U(\prod HA_i) \underset{\langle UH\pi_i\rangle}{\overset{\prod U\eta_{\alpha_i}\circ\varepsilon_{\prod A_i}}{\rightrightarrows}} \prod UHA_i$$

PROOF. We prove (1)$\Leftrightarrow$(2), (1)$\Leftrightarrow$(3), (2)$\Leftrightarrow$(4) and (2)$\Leftrightarrow$(5).

(1)⇔(2): This was proven in Theorem 2.4.6, as a corollary to Theorem 2.4.3.

(1)⇔(3): Let $\langle B,\, \beta\rangle$ be the product $\prod\langle A_i,\, \alpha_i\rangle$, with projections b_i. Then

$$\begin{aligned} Ub_i &= \varepsilon_{A_i} \circ U\eta_{\alpha_i} \circ Ub_i \\ &= \varepsilon_{A_i} \circ UH\pi_i \circ U \prod \eta_{\alpha_i} \\ &= \pi_i \circ \varepsilon_{\prod A_i} \circ U \prod \eta_{\alpha_i}, \end{aligned}$$

so, since each η_{α_i} is a regular mono, $\langle B,\, \beta\rangle$ is a regular subcoalgebra of $H\prod A_i$, with the composite

$$B \rightarrowtriangle UH\prod A_i \longrightarrow \prod A_i \longrightarrow A_i$$

a homomorphism.

Let $j\colon\langle D,\, \delta\rangle \rightarrowtriangle H\prod A_i$ be given and assume that, for each $i \in I$, the composite $\pi_i \circ \varepsilon_{\prod A_i} \circ Uj$ is a homomorphism. Then, because $\langle B,\, \beta\rangle$ is the product of the $\langle A_i,\, \alpha_i\rangle$'s, there is a unique homomorphism

$$k\colon\langle D,\, \delta\rangle \longrightarrow \langle B,\, \beta\rangle$$

such that, for each i,

$$Ub_i \circ Uk = \pi_i \circ \varepsilon_{\prod A_i} \circ Uj.$$

Using the previous calculation, we see

$$\pi_i \circ \varepsilon_{\prod A_i} \circ U \prod \eta_{\alpha_i} \circ Uk = \pi_i \circ \varepsilon_{\prod A_i} \circ Uj.$$

Hence, $\prod \eta_{\alpha_i} \circ k = j$, and $\langle D,\, \delta\rangle \le \langle B,\, \beta\rangle$, as desired.

(2)⇔(4): Suppose that $\langle B,\, \beta\rangle = \prod\langle A_i,\, \alpha_i\rangle$. By the proof of Theorem 2.4.6, we see that $\langle B,\, \beta\rangle = [P]_{\prod HA_i}$, where P is the pullback shown in (9).

In Theorem 2.2.11, we showed that the left hand square in Figure 8 is a pullback. Because the right hand square is just H applied to (9), it is also a pullback and so the rectangle is a pullback.

$$\begin{tikzcd} \langle B,\, \beta\rangle \arrow[r, tail, "b_1"] \arrow[d, tail, "b_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & HP \arrow[r, "Hp_1"] \arrow[d, tail, "Hp_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & H\prod A_i \arrow[d, tail, "H\prod U\eta_{\alpha_i}"] \\ H\prod A_i \arrow[r, tail, "\eta_{H\prod A_i}"'] & HUH\prod A_i \arrow[r, "H\langle UH\pi_i\rangle"'] & H\prod UHA_i \end{tikzcd}$$

FIGURE 8. $\prod\langle A_i,\, \alpha_i\rangle$ as a composite of two pullbacks in $\mathcal{E}_\Gamma$.

A simple calculation confirms that the composite along the bottom is $\prod \eta_{HA_i}$.

$$\begin{aligned}
H\langle UH\pi_i\rangle \circ \eta_{H\prod A_i} &= \langle HUH\pi_i \circ \eta_{H\prod A_i}\rangle \\
&= \langle \eta_{HA_i} \circ H\pi_i\rangle \\
&= \prod \eta_{HA_i} \circ H\langle \pi_i\rangle = \prod \eta_{HA_i}.
\end{aligned}$$

Conversely, if $\langle B,\, \beta\rangle$ is the pullback of $\prod \eta_{HA_i}$ along $\prod HU\eta_{\alpha_i}$, then there is a unique b_1 making the diagram in Figure 8 commute. Since the rectangle and the right hand square are pullbacks, so is the left hand square. Hence, $\langle B,\, \beta\rangle = [P]$ and thus $\langle B,\, \beta\rangle \cong \prod\langle A_i,\, \alpha_i\rangle$.

(2)⇔(5): Let P be the pullback of $\langle UH\pi_i\rangle$ along $\prod U\eta_{\alpha_i}$, and let E be the equalizer of $\langle UH\pi_i\rangle$ and $\prod U\eta_{\alpha_i} \circ \varepsilon_{\prod A_i}$. To show that $P \cong E$, it suffices to show that p_1 equalizes $\langle UH\pi_i\rangle$ and $\prod U\eta_{\alpha_i} \circ \varepsilon_{\prod A_i}$ (see Figure 9).

$$\begin{tikzcd}
P \arrow[r, tail, "p_1"] \arrow[d, "p_2"'] & UH\prod A_i \arrow[d, "\langle UH\pi_i\rangle"] \arrow[dl, "\varepsilon_{\prod A_i}"'] \\
\prod A_i \arrow[r, tail, bend left, "\prod U\eta_{\alpha_i}"] & \prod UHA_i \arrow[l, bend left, "\prod \varepsilon_{A_i}"]
\end{tikzcd}$$

FIGURE 9. $\prod\langle A_i,\, \alpha_i\rangle$ as an equalizer.

For this, we use the fact that, for every $i \in I$,

$$\pi_i \circ \varepsilon_{\prod A_i} = \varepsilon_{A_i} \circ UH\pi_i = \pi_i \circ \prod \varepsilon_{A_i} \circ \langle UH\pi_i\rangle,$$

and, hence, $\varepsilon_{\prod A_i} = \prod \varepsilon_{A_i} \circ \langle UH\pi_i\rangle$. Thus, we have

$$\begin{aligned}
\prod U\eta_{\alpha_i} \circ \varepsilon_{\prod A_i} \circ p_1 &= \prod U\eta_{\alpha_i} \circ \prod \varepsilon_{A_i} \circ \langle UH\pi_i\rangle \circ p_1 \\
&= \prod U\eta_{\alpha_i} \circ \prod \varepsilon_{A_i} \circ \prod U\eta_{\alpha_i} \circ p_2 \\
&= \prod U\eta_{\alpha_i} \circ p_2 = \langle UH\pi_i\rangle \circ p_1.
\end{aligned}$$

□

EXAMPLE 2.4.10. We consider the functor

$$\mathcal{P}_{\mathsf{fin}} : \mathbf{Set} \longrightarrow \mathbf{Set}$$

which takes a set A to the set of finite subsets of A. This functor preserves weak pullbacks and hence it preserves regular monos. We will calculate the product of two

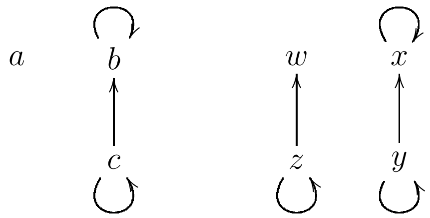

FIGURE 10. A graph representation of $\langle A,\, \alpha\rangle$ and $\langle X,\, \chi\rangle$.

simple $\mathcal{P}_{\mathsf{fin}}$-coalgebras. Although we do this in considerable detail here, in practice it is often quite simple. In particular, if a functor preserves pullbacks, then the product is easily calculated. This extended example will show how one actually uses many of the tools we've developed (embeddings into cofree coalgebras, the $[-]$ operator, etc.) to reason about coalgebras.

Recall that the product of coalgebras is constructed as a regular subcoalgebra of the product of the corresponding cofree coalgebras. Accordingly, in order to calculate this product, we first discuss the cofree $\mathcal{P}_{\mathsf{fin}}$-coalgebras. In order to ease the presentation, we use non-well-founded set theory. In the terms of **NWF**, it is easy to describe the cofree $\mathcal{P}_{\mathsf{fin}}$-coalgebra over A: It is the set UHA such that

$$UHA = A \times \mathcal{P}_{\mathsf{fin}}(UHA).$$

The structure map for this coalgebra, as usual, is the identity function. In particular, the final $\mathcal{P}_{\mathsf{fin}}$-coalgebra is the set of hereditarily finite (non-well-founded) sets.

We consider two uncomplicated coalgebras. Let A be the set $\{a, b, c$ and X the set $\{w, x, y, z\}$. We define the structure maps $\alpha : A \to \mathcal{P}_{\mathsf{fin}} A$ and $\chi : X \to \mathcal{P}_{\mathsf{fin}} X$ as follows:

$$\begin{aligned} \alpha(a) &= \emptyset, & \chi(w) &= \emptyset, \\ \alpha(b) &= \{b\}, & \chi(x) &= \{x\}, \\ \alpha(c) &= \{b, c\}, & \chi(y) &= \{x, y\}, \\ & & \chi(z) &= \{w, z\}. \end{aligned}$$

One calculates the units $\eta_\alpha : \langle A,\, \alpha\rangle \to HA$ and $\eta_\chi : \langle X,\, \chi\rangle \to HX$ thus:

$$\begin{aligned} \eta_\alpha(a) &= \langle a,\, \emptyset\rangle, & \eta_\chi(w) &= \langle w,\, \emptyset\rangle, \\ \eta_\alpha(b) &= \langle b,\, S_b\rangle, & \eta_\chi(x) &= \langle x,\, S_x\rangle, \\ \eta_\alpha(c) &= \langle c,\, S_c\rangle, & \eta_\chi(y) &= \langle y,\, S_y\rangle, \\ \eta_\chi(z) &= \langle z,\, S_z\rangle, \end{aligned}$$

where

$$\begin{aligned}
S_b &= \{\langle b, S_b\rangle\},\\
S_c &= \{\langle c, S_c\rangle, \langle b, S_b\rangle\},\\
S_x &= \{\langle x, S_x\rangle\},\\
S_y &= \{\langle x, S_x\rangle\langle y, S_y\rangle\},\\
S_z &= \{\langle w, \emptyset\rangle, \langle z, S_z\rangle\}.
\end{aligned}$$

The evident map

$$\langle e_A, e_X\rangle : U(HA \times HX) \cong UH(A \times X) \longrightarrow UHA \times UHX$$

is also easily described. The set $UH(A \times X)$ satisfies the equation

$$UH(A \times X) = A \times X \times \mathcal{P}_{\mathsf{fin}}(UH(A \times X)).$$

Let $\langle s, t, S\rangle \in A \times X \times \mathcal{P}_{\mathsf{fin}}(UH(A \times X))$. Then,

$$\langle e_A, e_X\rangle(\langle s, t, S\rangle) = \langle\langle s, S_A\rangle, \langle t, S_X\rangle\rangle,$$

where S_A is the image of S under e_A and S_X the image of S under e_X. In other words,

$$\begin{aligned}
S_A &= \mathcal{P}_{\mathsf{fin}} e_A(S),\\
S_X &= \mathcal{P}_{\mathsf{fin}} e_X(S).
\end{aligned}$$

Recall the definition of the set P from the proof of Theorem 2.4.3. From the definition of P as a pullback, we see that

$$P = \{\langle s, t, S\rangle \in UH(A \times X) \mid \langle e_A, e_X\rangle(\langle s, t, S\rangle) \in \mathsf{Im}(\eta_A \times \eta_X)\}.$$

Because A and X are such small sets, it is not difficult to calculate P directly.

Suppose, for some $t \in X$, $S \subseteq UH(A \times X)$, the triple $\langle a, t, S\rangle$ is in P. Then,

$$e_A(\langle a, t, S\rangle) = \langle a, \mathcal{P}_{\mathsf{fin}} e_A(S)\rangle = \eta_\alpha(a) = \langle a, \emptyset\rangle,$$

and so, S is empty. Since this entails that $\chi(t) = \emptyset$, we conclude that $t = w$. Similarly, the only triple of the form $\langle s, w, S\rangle$ is the triple $\langle a, w, \emptyset\rangle$.

Suppose that $\langle s, z, S\rangle$ is in P for some $s \in A$ and $S \subseteq UH(A \times X)$. Then, with a little work, one can show that $\langle a, w, \emptyset\rangle$ is in S. This entails that $a \in \alpha(s)$, yielding a contradiction. Thus, there is no triple of the form $\langle s, z, S\rangle$ in P.

Let $S \subseteq UH(A \times X)$ be given. Then, $\langle b, x, S\rangle$ is in P iff

$$\begin{aligned}
\mathcal{P}_{\mathsf{fin}} e_A(S) &= S_b = \{\langle b, S_b\rangle\},\\
\mathcal{P}_{\mathsf{fin}} e_X(S) &= S_x = \{\langle x, S_x\rangle\}.
\end{aligned}$$

These equations hold just in case $S \neq \emptyset$ and, for all $\langle u, v, T \rangle$ in S,

$$e_A(\langle u, v, T \rangle) = \langle b, S_b \rangle,$$
$$e_X(\langle u, v, T \rangle) = \langle x, S_x \rangle.$$

Thus, $\langle b, x, S \rangle \in P$ iff $S \neq \emptyset$ and, for all $\langle u, v, T \rangle \in S$, $u = b$, $t = x$ and $\langle u, v, T \rangle \in P$. We will use this fact to show that there is only one set S such that $\langle b, x, S \rangle \in P$.

We do this by using the principle of coinduction for the cofree coalgebra[5] $H(A \times X)$. We will show that, if $\langle b, x, S \rangle$ and $\langle b, x, S' \rangle$ are in P, then there is a coalgebraic relation

$$\langle R, \rho \rangle \in \mathsf{Rel}_{\mathcal{E}_{A \times X \times \mathcal{P}_{\mathsf{fin}}}}(H(A \times X))$$

relating $\langle b, x, S \rangle$ and $\langle b, x, S' \rangle$. Since $H(A \times X)$ is the final $A \times X \times \mathcal{P}_{\mathsf{fin}}$-coalgebra, we may conclude $\langle b, x, S \rangle = \langle b, x, S' \rangle$ (since equality is the largest relation on $H(A \times X)$).

We discuss relations on coalgebras and the related notion of bisimulation in more detail in Section 2.5. For now, it suffices to note that a relation R on $UH(A \times X)$ is the carrier for a relation on $H(A \times X)$ (in $\mathcal{E}_{A \times X \times \mathcal{P}_{\mathsf{fin}}}$) if, whenever $\langle s, t, T \rangle \, R \, \langle s', t', T' \rangle$, then

- $s = s'$,
- $t = t'$,
- for each $u \in T$, there is a $u' \in T'$ such that $u \, R \, u'$ and
- for each $u' \in T'$, there is a $u \in T$ such that $u \, R \, u'$.

Let R be the relation such that $\langle s, t, T \rangle \, R \, \langle s', t', T' \rangle$ holds iff

- $s = s' = b$,
- $t = t' = x$ and
- $\langle b, x, T \rangle$ and $\langle b, x, T' \rangle$ are in P.

Then, one may show that R is (the carrier of) a coalgebraic relation. Thus, there is at most one set S such that $\langle b, x, S \rangle \in P$.

Let $S_{b,x}$ satisfy the equation

$$S_{b,x} = \{\langle b, x, S_{b,x} \rangle\}.$$

A simple calculation verifies that $\langle b, x, S_{b,x} \rangle$ is in P.

A similar argument shows that $\langle b, y, S \rangle \in P$ iff $S = S_{b,y}$, where

$$S_{b,y} = \{\langle b, x, S_{b,x} \rangle, \langle b, y, S_{b,y} \rangle\}.$$

Also, $\langle c, x, S \rangle \in P$ iff $S = S_{c,x}$, where

$$S_{c,x} = \{\langle b, x, S_{b,x} \rangle, \langle c, x, S_{c,x} \rangle\}.$$

[5] One could also use the principle of coinduction for NWF to show that, if $\langle b, x, S \rangle$ and $\langle b, x, S' \rangle$ are in P, then S and S' are $\mathcal{P}$-bisimilar. The relation one defines to show $S \sim S'$ is more complicated, however.

Finally, we consider triples of the form $\langle c, y, S\rangle$. Such triples are in P just in case S satisfies the equations

$$\mathcal{P}_{\mathsf{fin}} e_A(S) = \{\langle b, S_b\rangle, \langle c, S_c\rangle\}, \tag{10}$$

$$\mathcal{P}_{\mathsf{fin}} e_X(S) = \{\langle x, S_x\rangle, \langle y, S_y\rangle\}. \tag{11}$$

Consider the set

$$S_{c,y} = \{\langle b, x, S_{b,x}\rangle, \langle c, y, S_{c,y}\rangle\}.$$

Then one can show that $S_{c,y}$ satisfies (10) and (11), so $\langle c, y, S_{c,y}\rangle$ is in P. However, the set

$$V = \{\langle c, x, S_{c,x}\rangle, \langle b, y, S_{b,y}\rangle\}$$

also satisfies (10) and (11). Indeed, there are many sets which satisfy these two equations: Any set S such that

$$V \subseteq S \subseteq V \cup S_{c,y} \text{ or } S_{c,y} \subseteq S \subseteq V \cup S_{c,y}.$$

satisfies (10) and (11), and one can show that these are the only sets which satisfy these equations.

Thus, we have characterized the set

$$P = \{\langle s, t, S\rangle \in UH(A \times B) \mid \langle e_A, e_X\rangle(\langle s, t, S\rangle) \in \mathsf{Im}(\eta_A \times \eta_X)\}.$$

Namely, P is the set

$$\{\langle a, w, \emptyset\rangle, \langle b, x, S_{b,x}\rangle, \langle b, y, S_{b,y}\rangle, \langle c, x, S_{c,x}\rangle\}$$

joined with the set

$$\{\langle c, y, S\rangle \mid V \subseteq S \subseteq V \cup S_{c,y} \text{ or } S_{c,y} \subseteq S \subseteq V \cup S_{c,y}\}.$$

By Corollary 2.4.4, the set P (with the projection π_3 as a structure map) is the product $\langle A, \alpha\rangle \times \langle X, \chi\rangle$. The projections are just the obvious projections:

$$\pi_\alpha(\langle s, t, S\rangle) = s,$$
$$\pi_\chi(\langle s, t, S\rangle) = t.$$

2.5. Bisimulations

We now turn our attention to bisimulations — relations on coalgebras which in some sense respect the coalgebraic structure. We had postponed our discussion of bisimulations until we had shown how one defines products of coalgebras. This allows a simple definition of regular relations in the category $\mathcal{E}_\Gamma$. We focus on regular relations for the same reasons that we restricted our attention to regular subobjects when defining subcoalgebras. Namely, since $\mathcal{E}_\Gamma$ inherits epi-regular mono factorizations, under our usual assumptions on $\mathcal{E}$ and Γ, the regular relations come with a richer set of construction principles. These are the relations that are well-behaved, both in $\mathcal{E}$ and in $\mathcal{E}_\Gamma$. A bisimulation will be a regular relation in $\mathcal{E}$ that is the image of a regular relation in $\mathcal{E}_\mathbb{G}$.

Compared to the definition of bisimulation we find in other works on coalgebras, the definition we adopt may seem a bit complicated. There are two reasons for the apparent complexity of our development. The first reason is that we stress the importance of regular relations and in the usual setting (coalgebras over **Set**), every relation is regular, thus removing the distinction. Secondly, in **Set**, one has the advantage of the axiom of choice. This simplifies the definition of bisimulation considerably (Theorem 2.5.8). So, one finds that, in the category **Set**, our definition coincides with the definition of bisimulation found in [**JR97**], [**BM96**], etc. The additional complexity of the definition of bisimulation found here seems a necessary effect of generalizing the setting in which we are interested.

Because the theory of bisimulations has not been well developed outside of $\mathbf{Set}_\Gamma$, we feel justified in offering an alternative definition for categories $\mathcal{E}_\Gamma$ which reduces to the familiar definition when $\mathcal{E} = \mathbf{Set}$. What one wants, however, is a compelling example of a category of coalgebras for which the two definitions differ, and for which the definition offered here is demonstrably preferable. Unfortunately, because of the results in Section 2.5.2 (which show that, if $\mathbb{G}$ preserves regular relations, then again Definition 2.5.4 reduces to the definition of bisimulation found elsewhere), such examples are difficult to come by. One would like to look at power object coalgebras over a topos which does not satisfy choice and see how the class of bisimulations discussed here differ from the class of coalgebraic relations preserved by U. This is an obvious area for future research.

Definition 2.5.1. Let $\mathcal{C}$ be a category with finite products. A relation R on A and B is a *regular relation* if the inclusion

$$R \rightarrowtail A \times B$$

is a regular mono.

Remark 2.5.2. The notion of a regular relation doesn't really require that $\mathcal{C}$ has finite products. One could say that R is a regular relation on A and B just in case, for every $C \in \mathcal{C}$, the map

$$\mathsf{Hom}(A, C) \times \mathsf{Hom}(B, C) \longrightarrow \mathsf{Hom}(R, C)$$

is a regular epi. See the definition of *regular epimorphic family* in [**BW85**] for details. We won't require that kind of generality here.

Remark 2.5.3. In a category in which every mono is regular (say, a category with a subobject classifier), every relation is regular.

Let $\mathcal{E}$ be almost co-regular and let Γ be a covarietor that preserves regular monos, with H the right adjoint to U. Let $\langle A,\, \alpha\rangle$ and $\langle B,\, \beta\rangle$ be Γ-coalgebras. Then there are two evident categories of regular relations to consider. On the one hand, there are the regular relations on A and B in $\mathcal{E}$, that is,

$$\mathsf{RegSub}_{\mathcal{E}}(A \times B).$$

On the other, there are the regular relations on $\langle A,\, \alpha\rangle$ and $\langle B,\, \beta\rangle$ in $\mathcal{E}_\Gamma$,

$$\mathsf{SubCoalg}(\langle A,\, \alpha\rangle \times \langle B,\, \beta\rangle),$$

which we abbreviate as $\mathsf{SubCoalg}(\alpha \times \beta)$. We define a functor

$$U_{\alpha,\beta}\colon \mathsf{SubCoalg}(\alpha \times \beta) \longrightarrow \mathsf{RegSub}(A \times B)$$

as follows: Given a regular relation $\langle R,\, \rho\rangle$ over $\langle A,\, \alpha\rangle$ and $\langle B,\, \beta\rangle$, with projections $r_1,\, r_2$, we factor

$$\langle U\pi_\alpha,\, U\pi_\beta\rangle \circ U\langle r_1,\, r_2\rangle$$

(i.e., we factor $\langle Ur_1,\, Ur_2\rangle$), as shown in Figure 11. In other words, $U_{\alpha,\beta} = \exists_{\langle U\pi_\alpha,\, U\pi_\beta\rangle} \circ U_{\alpha\times\beta}$.

$$\begin{tikzcd}
R \arrow[r, two heads] \arrow[d, tail] & U_{\alpha,\beta}\langle R,\, \rho\rangle \arrow[d, tail] \\
U(\alpha \times \beta) \arrow[r, "\langle U\pi_\alpha,\, U\pi_\beta\rangle"'] & A \times B
\end{tikzcd}$$

FIGURE 11. The definition of $U_{\alpha,\beta}$.

We define the category $\mathsf{Bisim}(\alpha, \beta)$ of bisimulations over $\langle A,\, \alpha\rangle$ and $\langle B,\, \beta\rangle$ to be the image of $U_{\alpha,\beta}$ in the category of regular relations over A and B (that is, $\mathsf{RegSub}(A \times B)$). Explicitly, $\mathsf{Bisim}(\alpha, \beta)$ is the full subcategory of $\mathsf{RegSub}(A \times B)$ consisting of $U_{\alpha,\beta}\langle R,\, \rho\rangle$ for $\langle R,\, \rho\rangle \in \mathsf{SubCoalg}(\alpha \times \beta)$.

$$\begin{tikzcd}
\mathsf{SubCoalg}(\alpha \times \beta) \arrow[rr, "U_{\alpha,\beta}"] \arrow[dr] & & \mathsf{RegSub}(A \times B) \\
& \mathsf{Bisim}(\alpha, \beta) \arrow[ur, tail] &
\end{tikzcd}$$

FIGURE 12. The definition of $\mathsf{Bisim}(\alpha, \beta)$.

Definition 2.5.4. Let $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ be Γ-coalgebras. A regular relation R on A and B is a *bisimulation on* $\langle A, \alpha\rangle$ *and* $\langle B, \beta\rangle$ just in case there is a regular relation $\langle S, \sigma\rangle$ on $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ such that R is the image of

$$S \rightarrowtail U(\alpha \times \beta) \longrightarrow A \times B\ .$$

In other words, R is a bisimulation just in case $R = \exists_{\langle U\pi_\alpha, U\pi_\beta\rangle} S$ for some relation $\langle S, \sigma\rangle$ on $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$.

This definition of bisimulation differs from the definition one finds in [**JR97**], etc. Typically, one defines a bisimulation over $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ as a relation R on A and B such that R can be augmented with a structure map making it a relation in $\mathcal{E}_\Gamma$. This simpler definition is well-suited for coalgebras over **Set**, but is not well-behaved when the base category does not satisfy the axiom of choice. For instance, the simpler definition does not, in general, define a class of relations closed under joins. Definition 2.5.4 is a proper generalization of the definition of bisimulation in ibid, since it reduces to the more familiar definition of bisimulation in the presence of choice, as the following theorem shows (it also reduces to the simpler definition if Γ preserves regular relations — see Corollary 2.5.27).

The next few theorems give standard examples of bisimulations, which can be found in most introductions to coalgebras. One important construction of bisimulations does not seem to hold in this setting generally, however. It is apparently not the case that if R and S are composable bisimulations, then $R \circ S$ is a bisimulation. From [**JR97**], we have a proof that $R \circ S$ is a bisimulation, given that $\mathcal{E}$ satisfies the axiom of choice. In Section 4.2.6, we prove (using the internal logic of $\mathcal{E}_\Gamma$ and $\mathcal{E}$ developed in Chapter 4) that $R \circ S$ is a bisimulation if Γ preserves regular relations. In both of the cases in which we have proofs that bisimulations compose, the bisimulations consist of those relations in $\mathcal{E}$ which can be augmented with structure maps, making the projections homomorphisms. More general results would be nice, but the situation is unclear.

Theorem 2.5.5. *For any coalgebra* $\langle A, \alpha\rangle$*,* Δ_A *is a bisimulation.*

Proof. $U_{\alpha,\alpha}\Delta_\alpha$ is the image of

$$\langle U\pi_1, U\pi_2\rangle \circ U\langle \mathsf{id}_\alpha, \mathsf{id}_\alpha\rangle = \langle U\,\mathsf{id}_\alpha, U\,\mathsf{id}_\alpha\rangle = \langle \mathsf{id}_A, \mathsf{id}_A\rangle,$$

and so $U_{\alpha,\alpha}\Delta_\alpha = \Delta_A$. □

Theorem 2.5.6. *Let* $f : \langle A, \alpha\rangle \to \langle B, \beta\rangle$ *be a* Γ*-homomorphism. Then the graph of* Uf *is a bisimulation.*

Proof. The graph of f in $\mathcal{E}_\Gamma$ is the relation $\langle\langle A, \alpha\rangle, \mathsf{id}_\alpha, f\rangle$. Hence, $U_{\alpha,\beta}\,\mathsf{graph}(f)$ is the image of

$$\langle U\pi_\alpha, U\pi_\beta\rangle \circ U\langle \mathsf{id}_\alpha, f\rangle = \langle \mathsf{id}_A, Uf\rangle.$$

Therefore, $U_{\alpha,\beta}\,\mathsf{graph}(f) = \mathsf{graph}(Uf)$. $\square$

The next theorem is well-known, first appearing in [**Rut96**]. Since our definition of bisimulations include all those relations which are the carrier for some subcoalgebra of $\alpha \times \beta$, the result also holds in our setting. We include the proof nonetheless.

THEOREM 2.5.7. *If Γ preserves weak pullbacks, then for any pair of homomorphisms*

$$f : \langle A,\, \alpha\rangle \longrightarrow \langle B,\, \beta\rangle\,,$$
$$g : \langle C,\, \gamma\rangle \longrightarrow \langle B,\, \beta\rangle\,,$$

the pullback of f along g (properly, Uf along Ug) is a bisimulation.

PROOF. Let E be the pullback of f along g, as shown in Figure 13. Since, by assumption, Γ preserves weak pullbacks, the top face is a weak pullback. Hence, there is a structure map $\varepsilon : E \to \Gamma E$ making the two projections homomorphisms. Therefore, the inclusion $E \rightarrowtail A \times C$ factors through $U(\alpha \times \beta) \to A \times B$ and thus $\langle E,\, \varepsilon\rangle$ is a regular relation in $\mathcal{E}_\Gamma$. It is easy to verify that $U_{\alpha,\beta}\langle E,\, \varepsilon\rangle = E$.

$$\begin{tikzcd}
\Gamma E \arrow[rr] \arrow[dr] & & \Gamma C \arrow[dr] & \\
& \Gamma A \arrow[rr] & & \Gamma B \\
E \arrow[uu, dotted] \arrow[rr] \arrow[dr] & & C \arrow[uu] \arrow[dr] & \\
& A \arrow[uu] \arrow[rr] & & B \arrow[uu]
\end{tikzcd}$$

FIGURE 13. Pullbacks of homomorphisms are bisimulations.

$\square$

THEOREM 2.5.8. *Let $\langle A,\, \alpha\rangle$ and $\langle B,\, \beta\rangle$ be Γ-coalgebras and suppose that $\mathcal{E}$ satisfies the axiom of choice. Then a relation R on A and B is a bisimulation iff there is a structure map*

$$\rho : R \longrightarrow \Gamma R$$

such that the projections r_1 and r_2 are Γ-homomorphisms.

PROOF. Clearly, if R has a structure map making r_1 and r_2 homomorphisms, then R is a bisimulation.

Suppose that R is a bisimulation. Let $\langle S, \sigma\rangle \in \mathsf{SubCoalg}(\alpha \times \beta)$ such that $R = U_{\alpha,\beta}\langle S, \sigma\rangle$, with $p\colon S \twoheadrightarrow R$ the (necessarily regular) epi part of the factorization, as shown in Figure 14, and i the right inverse of p. Then it is easy to see that $\Gamma p \circ \sigma \circ i$ suffices as the desired structure map. $\square$

$$\begin{tikzcd}
\Gamma S \arrow[r, "\Gamma p"] & \Gamma R \arrow[r, tail] & \Gamma A \times \Gamma B \\
S \arrow[u, "\sigma"] \arrow[r, bend left, two heads, "p"] & R \arrow[u, dotted] \arrow[l, bend left, "i"] \arrow[r, tail] & A \times B \arrow[u, "\alpha\times\beta"']
\end{tikzcd}$$

FIGURE 14. Definition of a structure map for a bisimulation, given choice.

2.5.1. The right adjoint to $U_{\alpha,\beta}$. In Section 2.2.1, we saw that the subcoalgebra forgetful functor U_α has a right adjoint. We generalize that result to the functor $U_{\alpha,\beta}$ here.

REMARK 2.5.9. In what follows, we write $\alpha \times \beta$ as an abbreviation for $\langle A, \alpha\rangle \times \langle B, \beta\rangle$. This is not to be confused with the morphism

$$\alpha \times \beta\colon A \times B \longrightarrow \Gamma A \times \Gamma B$$

in $\mathcal{E}$.

THEOREM 2.5.10. *$U_{\alpha,\beta}$ has a right adjoint.*

PROOF. By definition,

$$U_{\alpha,\beta} = \exists_{\langle U\pi_\alpha,\, U\pi_\beta\rangle} \circ U_{\alpha\times\beta}.$$

Since $U_{\alpha\times\beta} \dashv [-]_{\alpha\times\beta}$ and $\exists_{\langle U\pi_\alpha,\, U\pi_\beta\rangle} \dashv \langle U\pi_\alpha, U\pi_\beta\rangle^*$ (pullback along $\langle U\pi_\alpha, U\pi_\beta\rangle$), the composite

$$[-]_{\alpha,\beta} = [-]_{\alpha\times\beta} \circ \langle U\pi_\alpha, U\pi_\beta\rangle^*$$

is a right adjoint to $U_{\alpha,\beta}$. $\square$

COROLLARY 2.5.11. *$U_{\alpha,\beta}$ preserves colimits.*

THEOREM 2.5.12. *Given $R \leq A\times B$, $A = U\langle A, \alpha\rangle$, $B = U\langle B, \beta\rangle$, the coalgebraic relation $[R]_{\alpha,\beta}$ is the pullback shown below, where the arrow on the bottom is the adjoint transpose of $\langle U\pi_\alpha, U\pi_\beta\rangle$.*

$$\begin{tikzcd}
{[R]_{\alpha,\beta}} \arrow[r, tail, "f"] \arrow[d, tail, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & HR \arrow[d, tail, "H\langle r_1, r_2\rangle"] \\
\alpha \times \beta \arrow[r, tail] & H(A \times B)
\end{tikzcd}$$

$$\begin{tikzcd}
{[p^*R]} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & Hp^*R \arrow[r] \arrow[d, tail] & HR \arrow[d, tail] \\
\alpha \times \beta \arrow[r, "\eta_{\alpha\times\beta}"'] & HU(\alpha\times\beta) \arrow[r, "Hp"'] & H(A\times B)
\end{tikzcd}$$

FIGURE 15. Alternate definition of $[-]_{\alpha,\beta}$

PROOF. Let $p = \langle U\pi_1,\, U\pi_2\rangle : U(\alpha\times\beta) \to A\times B$. By Theorem 2.2.11 (3), $[p^*R]$ is the pullback on the left hand square of Figure 15. The right hand square is also a pullback, since H preserves pullbacks. Hence, the composite is a pullback. $\square$

The adjoint functors $U_{\alpha,\beta}$ and $[-]_{\alpha,\beta}$ give rise to a monad on $\mathsf{RegSub}(A\times B)$ and a comonad on $\mathsf{SubCoalg}(\alpha\times\beta)$, that is, an interior operator $\Box_{\alpha,\beta} = U_{\alpha,\beta}[-]_{\alpha,\beta}$ and a closure operator $\bigtriangledown_{\alpha,\beta} = [-]_{\alpha,\beta}U_{\alpha,\beta}$. In the case of subcoalgebras, the comonad $[-]_\alpha U_\alpha$ is just the identity on $\mathsf{SubCoalg}(\langle A,\, \alpha\rangle)$, but for relations, this is not generally the case, as the following example shows. Instead, the closure of a coalgebraic relation $\langle R,\, \rho\rangle$ on $\langle A,\, \alpha\rangle$ and $\langle B,\, \beta\rangle$ is the largest relation $\langle S,\, \sigma\rangle$ such that

$$U_{\alpha,\beta}\langle R,\, \rho\rangle = U_{\alpha,\beta}\langle S,\, \sigma\rangle.$$

We return to a discussion of $\Box_{\alpha,\beta}$ in Section 2.5.2.

EXAMPLE 2.5.13. Consider again the $\mathcal{P}_{\mathsf{fin}}$-coalgebras $\langle A,\, \alpha\rangle$ and $\langle X,\, \chi\rangle$ from Example 2.4.10. Recall that A is the set $\{a, b, c\}$ and X the set $\{w, x, y, z\}$. The structure maps $\alpha : A \to \mathcal{P}_{\mathsf{fin}}A$ and $\chi : X \to \mathcal{P}_{\mathsf{fin}}X$ are given by:

$$\begin{aligned}
\alpha(a) &= \emptyset, & \chi(w) &= \emptyset,\\
\alpha(b) &= \{b\}, & \chi(x) &= \{x\},\\
\alpha(c) &= \{b, c\}, & \chi(y) &= \{x, y\},\\
& & \chi(z) &= \{w, z\}.
\end{aligned}$$

We calculated their product as the set

$$\{\langle a,\, w,\, \emptyset\rangle, \langle b,\, x,\, S_{b,x}\rangle, \langle b,\, y,\, S_{b,y}\rangle, \langle c,\, x,\, S_{c,x}\rangle\}$$

joined with the set

$$\{\langle c,\, y,\, S\rangle \mid V \subseteq S \subseteq V \cup S_{c,y} \text{ or } S_{c,y} \subseteq S \subseteq V \cup S_{c,y}\},$$

where

$$
\begin{aligned}
S_{b,x} &= \{\langle b,\, x,\, S_{b,x}\rangle\},\\
S_{b,y} &= \{\langle b,\, x,\, S_{b,x}\rangle, \langle b,\, y,\, S_{b,y}\rangle\},\\
S_{c,x} &= \{\langle b,\, x,\, S_{b,x}\rangle, \langle c,\, x,\, S_{c,x}\rangle\},\\
S_{c,y} &= \{\langle b,\, x,\, S_{b,x}\rangle, \langle c,\, y,\, S_{c,y}\rangle\},\\
V &= \{\langle c,\, x,\, S_{c,x}\rangle, \langle b,\, y,\, S_{b,y}\rangle\}.
\end{aligned}
$$

We consider a relation $\langle R,\, \rho\rangle$ on $\langle A,\, \alpha\rangle$ and $\langle X,\, \chi\rangle$ where

$$R = \{\langle a,\, w,\, \emptyset\rangle, \langle b,\, x,\, S_{b,x}\rangle, \langle b,\, y,\, S_{b,y}\rangle, \langle c,\, x,\, S_{c,x}\rangle, \langle c,\, y,\, S_{c,y}\rangle\}.$$

The structure map ρ on R is the projection π_3.

One sees that $U_{\alpha,\chi}R$ is the relation $\{\langle a,\, w\rangle, \langle b,\, x\rangle, \langle b,\, y\rangle, \langle c,\, x\rangle, \langle c,\, y\rangle\}$. In other words, $U_{\alpha,\chi}R$ is the largest bisimulation on $\langle A,\, \alpha\rangle$ and $\langle X,\, \chi\rangle$. Consequently,

$$\nabla_{\alpha,\chi}\langle R,\, \rho\rangle = \alpha \times \chi \gneqq \langle R,\, \rho\rangle.$$

The following observation is a standard fact about Galois correspondences.

THEOREM 2.5.14. *The following posets are isomorphic:*

$$\mathsf{Fix}(\Box_{\alpha,\beta}) \cong \mathsf{Fix}(\nabla_{\alpha,\beta}) \cong \mathsf{Bisim}(\alpha, \beta),$$

(where Fix *is the poset of fixed points of the operator).*

Let $f : \langle A,\, \alpha\rangle \to \langle B,\, \beta\rangle$ be given. We saw, in Corollary 2.2.9 that Uf factors through a subobject P of B just in case f factors through $[P]$. We prove an analogous result here for pairs of homomorphisms into $\alpha \times \beta$. First, we prove a lemma.

LEMMA 2.5.15. *Let*

$$
\begin{aligned}
f &: \langle C,\, \gamma\rangle \longrightarrow \langle A,\, \alpha\rangle,\\
g &: \langle C,\, \gamma\rangle \longrightarrow \langle B,\, \beta\rangle
\end{aligned}
$$

be Γ*-homomorphisms. Then*

$$U_{\alpha,\beta}\,\mathsf{Im}\langle f,\, g\rangle = \mathsf{Im}\langle Uf,\, Ug\rangle.$$

PROOF. We use the facts that

$$
\begin{aligned}
\langle Uf,\, Ug\rangle &= \langle U\pi_\alpha,\, U\pi_\beta\rangle \circ U\langle f,\, g\rangle,\\
U\,\mathsf{Im}\langle f,\, g\rangle &= \mathsf{Im}\,U\langle f,\, g\rangle.
\end{aligned}
$$

The commutative diagram in Figure 16 completes the proof. $\square$

$$\begin{tikzcd}
\bullet \arrow[r, two heads] \arrow[dr, "U\langle f,\, g\rangle"'] & U\,\mathsf{Im}\langle f,\, g\rangle \arrow[r, two heads] \arrow[d, tail] & U_{\alpha,\beta}\,\mathsf{Im}\langle f,\, g\rangle \arrow[d, tail] \\
& U(\alpha \times \beta) \arrow[r, "\langle U\pi_\alpha,\, U\pi_\beta\rangle"'] & A \times B
\end{tikzcd}$$

FIGURE 16. $U_{\alpha,\beta}$ commutes with Im.

THEOREM 2.5.16. *Let*

$$f : \langle C,\, \gamma\rangle \longrightarrow \langle A,\, \alpha\rangle\,,$$
$$g : \langle C,\, \gamma\rangle \longrightarrow \langle B,\, \beta\rangle$$

be Γ-homomorphisms and R a relation on A and B. Then $\langle f,\, g\rangle$ factors through $[R]_{\alpha,\beta}$ just in case $\langle Uf,\, Ug\rangle$ factors through R.

PROOF.

$$\mathsf{Im}\langle f,\, g\rangle \leq [R]_{\alpha,\beta} \text{ iff } U_{\alpha,\beta}\,\mathsf{Im}\langle f,\, g\rangle \leq R \text{ iff } \mathsf{Im}\langle Uf,\, Ug\rangle \leq R.$$

□

COROLLARY 2.5.17. *Let the left hand square of Figure 17 be a pullback. Then the right hand square is also a pullback.*

$$\begin{tikzcd}
\bullet \arrow[dr, dotted] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "k"'] & & \\
& P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[dr, dotted] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "k"'] & & \\
& {[P]_{\alpha,\beta}} \arrow[r] \arrow[d] & \langle B,\, \beta\rangle \arrow[d, "g"] \\
& \langle A,\, \alpha\rangle \arrow[r, "f"'] & \langle C,\, \gamma\rangle
\end{tikzcd}$$

FIGURE 17. $[P]$ is a pullback.

PROOF. Let h and k be homomorphisms making the right hand square commute. Then the left hand square, which is just the image of the right hand square under U, also commutes, and so there is a unique factorization of $\langle h,\, k\rangle$ through P. Apply Theorem 2.5.16 to conclude that $\langle h,\, k\rangle$ factors through $[P]$. □

REMARK 2.5.18. Example 2.5.13 also shows that the functor $U_{\alpha,\beta}$ is not generally full. In this example, $U_{\alpha,\beta}(\alpha \times \beta) \leq U_{\alpha,\beta}\langle R,\, \rho\rangle$, but $\alpha \times \beta \not\leq \langle R,\, \rho\rangle$. This is a difference between the subcoalgebra functor U_α and the bisimulation functor $U_{\alpha,\beta}$.

The functor $[-]_{\alpha,\beta}$ is a natural transformation between contravariant bifunctors. In order to make that precise, we define the functors $\mathsf{RegRel}(\langle A,\,\alpha\rangle,\langle B,\,\beta\rangle)$ (abbreviated $\mathsf{RegRel}(\alpha,\beta)$) and $\mathsf{RegRel}(A,B)$ as bifunctors. Their definition is clear from the preceding discussion. Namely,

$$\mathsf{RegRel}(\alpha,\beta) = \mathsf{SubCoalg}(\alpha\times\beta) \text{ and}$$
$$\mathsf{RegRel}(A,B) = \mathsf{RegSub}(A\times B).$$

Thus, the effect of $\mathsf{RegRel}_{\mathcal{E}_\Gamma}$, say, on a pair of maps

$$f\colon\langle A,\,\alpha\rangle \longrightarrow \langle C,\,\gamma\rangle$$
$$g\colon\langle B,\,\beta\rangle \longrightarrow \langle D,\,\delta\rangle$$

is a functor

$$\mathsf{RegRel}(f,g)\colon \mathsf{RegRel}(\gamma,\delta) \longrightarrow \mathsf{RegRel}(\alpha,\beta)\,.$$

Namely, it takes a relation $\langle R,\,\rho\rangle$ on $\langle C,\,\gamma\rangle$ and $\langle D,\,\delta\rangle$ to the pullback shown below.

$$\begin{tikzcd}
(f\times g)^*\langle R,\,\rho\rangle \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \langle R,\,\rho\rangle \arrow[d, tail] \\
\alpha\times\beta \arrow[r, "f\times g"'] & \gamma\times\delta
\end{tikzcd}$$

THEOREM 2.5.19. $[-]\colon \mathsf{RegRel}_{\mathcal{E}} \circ U\times U \Rightarrow \mathsf{RegRel}_{\mathcal{E}_\Gamma}$ *is natural. I.e., for every pair of maps, f and g, as above,*

$$[-]\circ(Uf\times Ug)^* = (f\times g)^*\circ[-].$$

PROOF. Let $f\colon\langle A,\,\alpha\rangle\to\langle C,\,\gamma\rangle$ and $g\colon\langle B,\,\beta\rangle\to\langle D,\,\delta\rangle$ be given. Let R be a regular relation over C and D and $S = (Uf\times Ug)^*R$. We will show that $[S]_{\alpha,\beta} = (f,g)^*[R]_{\gamma,\delta}$.

$$\begin{tikzcd}
{[S]_{\alpha,\beta}} \arrow[rr] \arrow[dr] \arrow[dd, tail] & & HS \arrow[dr] \arrow[dd, tail] & \\
& {[R]_{\gamma,\delta}} \arrow[rr] \arrow[dd, tail] & & HR \arrow[dd, tail] \\
\alpha\times\beta \arrow[rr] \arrow[dr, "f\times g"'] & & HA\times HB \arrow[dr, "HUf\times HUg"] & \\
& \gamma\times\delta \arrow[rr, "\eta_\gamma\times\eta_\delta"'] & & HC\times HD
\end{tikzcd}$$

FIGURE 18. $[-]$ is natural.

In Figure 18, the front and rear faces are pullbacks by Theorem 2.5.12. The right hand face is also a pullback, since H preserves pullbacks. The bottom commutes by naturality. The arrow

$$[S]_{\alpha,\beta} \longrightarrow [R]_{\gamma,\delta}$$

is the unique map making the top and left hand squares commute (due to the pullback in front).

Because the composite of the left and front faces is a pullback, and so is the front face itself, we see that the left face is a pullback. $\square$

For each pair of maps,

$$f : \langle A,\, \alpha\rangle \longrightarrow \langle C,\, \gamma\rangle$$
$$g : \langle B,\, \beta\rangle \longrightarrow \langle D,\, \delta\rangle$$

the functor $(f \times g)^*$ (i.e., $\mathsf{RegRel}(f, g)$) has a left adjoint

$$\exists_{f,g} : \mathsf{RegRel}(\alpha, \beta) \longrightarrow \mathsf{RegRel}(\gamma, \delta)\,.$$

Namely, given a regular relation $\langle R,\, \rho\rangle$ on $\langle A,\, \alpha\rangle$ and $\langle B,\, \beta\rangle$, we take the epi-regular mono factorization shown below.

$$\begin{tikzcd}
\langle R,\, \rho\rangle \arrow[r, two heads] \arrow[d, tail] & \exists_{f,g}\langle R,\, \rho\rangle \arrow[d, tail] \\
\alpha \times \beta \arrow[r, "f\times g"'] & \gamma \times \delta
\end{tikzcd}$$

The same fact holds in $\mathcal{E}$ as well. That is, for any pair of maps h and k, the pullback functor $(h \times k)^*$ has a left adjoint, $\exists_{h,k}$.

$$\begin{tikzcd}
\mathsf{RegRel}(\alpha, \beta) \arrow[r, shift left, "\exists_{f,g}"] \arrow[r, phantom, "\perp"] \arrow[d, shift right, "U_{\alpha,\beta}"'] \arrow[d, phantom, "\dashv"] & \mathsf{RegRel}(\gamma, \delta) \arrow[l, shift left, "(f\times g)^*"] \arrow[d, shift right, "U_{\gamma,\delta}"'] \arrow[d, phantom, "\dashv"] \\
\mathsf{RegRel}(A, B) \arrow[r, shift left, "\exists_{Uf,Ug}"] \arrow[r, phantom, "\perp"] \arrow[u, shift right, "{[-]_{\alpha,\beta}}"'] & \mathsf{RegRel}(C, D) \arrow[l, shift left, "(Uf\times Ug)^*"] \arrow[u, shift right, "{[-]_{\gamma,\delta}}"']
\end{tikzcd}$$

FIGURE 19. $U_{\alpha,\beta}$ commutes with $\exists$

The following corollary is found in [**JR97**, Lemma 5.3], where it is proved for coalgebras over **Set**. Of course, the proof offered here differs inasmuch as it uses our definition of bisimulation, but the basic approach is the same.

Name	Category	Description
Subalgebra	SubAlg	A subobject of $\langle A, \alpha\rangle$ preserved by U.
Pre-congruence	PreCong	A relation on $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ preserved by U
Congruence	Cong	A pre-congruence equivalence relation
Subcoalgebra	SubCoalg	A regular subobject of $\langle A, \alpha\rangle$ (necessarily preserved by U).
Bisimulation	Bisim	The image of a regular relation over $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$
Bisimulation equivalence	BisimEq	A bisimulation which is an equivalence relation

TABLE 1. A summary of predicates and relations.

COROLLARY 2.5.20. *$U_{\alpha,\beta}$ commutes with $\exists$. In other words, for any pair of homomorphisms $f:\langle C, \gamma\rangle \twoheadrightarrow \langle A, \alpha\rangle$, $g:\langle C, \gamma\rangle \twoheadrightarrow \langle B, \beta\rangle$, the image of $\langle f, g\rangle$ is a bisimulation over $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$.*

PROOF. The right adjoints in Figure 19 commute by Theorem 2.5.19, and so the left adjoints also commute. Thus,

$$\exists_{Uf,Ug} \circ U_{\alpha,\beta} = U_{\gamma,\delta} \circ \exists_{f,g}.$$

$\square$

2.5.2. $\Box_{\alpha,\beta}$ and relation-preserving functors. In Section 2.2.2, we saw that the operator $\Box_\alpha = U_\alpha \circ [-]_\alpha$ is an **S4** modal operator. The situation for the analogous bisimulation operator is more difficult. The operator

$$\Box_{\alpha,\beta} : \mathsf{RegRel}(A, B) \longrightarrow \mathsf{RegRel}(A, B)$$

defined by $\Box_{\alpha,\beta} = U_{\alpha,\beta}[-]_{\alpha,\beta}$ easily satisfies certain of the properties of **S4** operators, namely

- $\Box$ is monotone;
- $\Box$ is deflationary;
- $\Box$ is idempotent.

These properties are satisfied by any comonad on a poset. Nonetheless, it is not clear that $\Box$ is a *normal* modal operator, that is, that

$$\Box_{\alpha,\beta} R \wedge \Box_{\alpha,\beta} S \leq \Box_{\alpha,\beta}(R \wedge S). \tag{12}$$

Indeed, even over **Set**, bisimulations need not be closed under finite meets, and so $\Box$ need not be normal. Worse, even if $\Box$ does preserve binary meets, it does not generally preserve $\top$, so $\Box$ simply won't be **S4** typically.

In this section, we give sufficient conditions that $\Box$ preserves binary meets, namely that the endofunctor Γ preserve regular relations. This is a fairly strong condition and is not met by some functors of interest. In the following example, we will show that the $\Box$ operator for $\mathcal{P}_{\mathsf{fin}}$-coalgebras is not normal.

EXAMPLE 2.5.21. Consider the finite powerset functor and the coalgebra $\langle A,\, \alpha\rangle$ represented by the graph in Figure 20. Let

$$\begin{aligned} R &= \{\langle a,\, a\rangle, \langle b,\, b\rangle, \langle c,\, c\rangle\},\\ S &= \{\langle a,\, a\rangle, \langle b,\, c\rangle, \langle c,\, b\rangle\}. \end{aligned}$$

Then $\Box R = R$, $\Box S = S$ and $\Box(R \wedge S) = \Box\{\langle a,\, a\rangle\} = \emptyset$. (Thanks to Tobias Schröder for this example.)

$$\begin{tikzcd} & a \arrow[dl] \arrow[dr] & \\ b & & c \end{tikzcd}$$

FIGURE 20. $\mathcal{P}_{\mathsf{fin}}$-bisimulations are not closed under $\wedge$.

As we will see, if Γ preserves regular relations, then the category of bisimulations $\mathsf{Bisim}(\alpha, \beta)$ inherits much of its structure from the category $\mathsf{RegSub}(A \times B)$ of relations in $\mathcal{E}$. In fact, in this case, the category $\mathsf{Bisim}(\alpha, \beta)$ is isomorphic to $\mathsf{SubCoalg}(\alpha \times \beta)$ and is a full subcategory of $\mathsf{RegSub}(A \times B)$, with the inclusion a complete Heyting algebra homomorphism. Such a close connection between these three categories requires a correspondingly strong assumption on Γ.

From [**GS01**], we learn that a functor Γ preserves pullbacks iff Γ preserves weak pullbacks and mono 2-sources (i.e., binary relations). Indeed, the same claim holds if we replace mono 2-sources with regular mono 2-sources (regular relations). We include this and other proofs from ibid here, replacing mono 2-sources with regular relations.

DEFINITION 2.5.22. A functor Γ *preserves regular relations* if, for every regular relation $\langle R,\, r_1,\, r_2\rangle$ on X and Y, the triple $\langle \Gamma R,\, \Gamma r_1,\, \Gamma r_2\rangle$ is a regular relation on ΓX, ΓY, i.e., $\langle \Gamma r_1,\, \Gamma r_2\rangle$ is a regular mono.

As Gumm and Schröder showed, it is sufficient that Γ take binary products to (in our setting, regular) relations.

LEMMA 2.5.23. Γ *preserves regular relations iff, for every X, Y,*

$$\langle \Gamma(X \times Y),\, \Gamma\pi_X,\, \Gamma\pi_Y\rangle$$

is a regular relation.

PROOF. Clearly if Γ preserves regular relations, then it preserves the regular relation $X \times Y$. Suppose, conversely, that for every X, Y, $\Gamma(X \times Y)$ is a regular relation, i.e., $\langle \Gamma\pi_X, \Gamma\pi_Y \rangle$ is a regular mono. Then, for any regular relation $R \rightarrowtail X \times Y$, the composite

$$\Gamma R \rightarrowtail \Gamma(X \times Y) \rightarrowtail \Gamma X \times \Gamma Y$$

is a regular relation. ☐

THEOREM 2.5.24. *Γ preserves pullbacks iff Γ preserves weak pullbacks and regular relations.*

PROOF. If Γ preserves pullbacks, then Γ takes the pullback square

$$\begin{tikzcd}
X \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\
X \arrow[r] & 1
\end{tikzcd}$$

to a pullback, so $\langle \Gamma\pi_1, \Gamma\pi_2 \rangle$ is a regular mono. Apply Theorem 2.5.23 to conclude that Γ preserves regular relations.

For the converse, notice that pullbacks are both regular relations and weak pullbacks, and that a weak pullback which is a regular relation is also a pullback. ☐

On the one hand, as the following theorems show, preservation of regular relations is the "right" condition to ensure well-behaved bisimulations. On the other hand, preservation of regular relations is an unfortunately strong condition, not satisfied by many functors of interest (such as $\mathcal{P}_{\mathsf{fin}}$). Nonetheless, there seems to be no reasonable middle ground. If one wants $\Box$ to be well-behaved as a modal operator (although, even here, we will typically not preserve the final subobject), then one must restrict interest to pullback-preserving functors (or some similarly suitable domain).

THEOREM 2.5.25. *If Γ preserves regular relations, then U preserves regular relations. In other words, for any relation $\langle\langle R, \rho\rangle, r_1, r_2\rangle$ on $\langle A, \alpha\rangle$, $\langle B, \beta\rangle$,*

$$U_{\alpha,\beta}\langle\langle R, \rho\rangle, r_1, r_2\rangle = \langle R, r_1, r_2\rangle.$$

PROOF. It suffices to show that, for every pair of coalgebras $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$, $U(\alpha \times \beta) \rightarrow A \times B$ is a regular mono. We sketch how to do that here, leaving details to the reader.

First, one shows that U creates epi-regular mono 2-source factorizations. That is, for each pair of homomorphisms,

$$f : \langle C, \gamma\rangle \longrightarrow \langle A, \alpha\rangle,$$
$$g : \langle C, \gamma\rangle \longrightarrow \langle B, \beta\rangle,$$

there is a unique epi $p{:}\langle C,\,\gamma\rangle \twoheadrightarrow \langle D,\,\delta\rangle$ and pair

$$h{:}\langle D,\,\delta\rangle \longrightarrow \langle A,\,\alpha\rangle\,,$$
$$k{:}\langle D,\,\delta\rangle \longrightarrow \langle B,\,\beta\rangle\,,$$

such that $h \circ p = f$ and $k \circ p = g$ and $\langle Uh,\,Uk\rangle$ is regular mono in $\mathcal{E}$.

Say that a regular relation $\langle S,\,s_1,\,s_2\rangle$ on A and B is α,β-invariant if there is a structure map $\sigma{:}S \to \Gamma S$ such that s_1 and s_2 are homomorphisms. Let R be the join of all α,β-invariant relations S. Using the fact about epi-regular mono 2-source factorizations above, one can show that R is itself α,β-invariant, with unique structure map $\rho{:}R \to \Gamma R$. Moreover, one can show that the coalgebra $\langle R,\,\rho\rangle$ is, in fact, the product of $\langle A,\,\alpha\rangle$ and $\langle B,\,\beta\rangle$. Hence, $U(\alpha \times \beta)$ is a regular relation over A and B. $\square$

The categories $\mathsf{RegRel}(\alpha,\beta)$ and $\mathsf{RegRel}(A,B)$ are both complete Heyting algebras, since they are simply categories of subobjects of $\alpha \times \beta$ and $A \times B$, respectively. The forgetful functor $U_{\alpha,\beta}$ is not, however, a Heyting algebra homomorphism in general, since it does not preserves meets (Example 2.5.21). By Theorem 2.2.6, we know that $U_{\alpha\times\beta}$ preserves meets, but the functor

$$\exists_{\langle U\pi_1,\,U\pi_2\rangle}{:}\mathsf{RegSub}(U(\alpha \times \beta)) \longrightarrow \mathsf{RegSub}(A \times B)$$

generally does not preserve meets. Assuming that Γ preserves regular relations, however, $\exists_{\langle U\pi_1,\,U\pi_2\rangle}$ does preserve meets, and hence we have the following corollary.

Corollary 2.5.26. *If Γ preserves regular relations, then $\Box$ distributes over $\wedge$. In other words, if Γ preserves regular relations, the meet (in $\mathcal{E}$) of two bisimulations is again a bisimulation.*

Proof. Let $\langle A,\,\alpha\rangle$ and $\langle B,\,\beta\rangle$ be given, R and S be relations over A and B, and suppose Γ preserves regular relations. Then, by Theorem 2.5.25, U also preserves regular relations and, hence,

$$p = \langle U\pi_1,\,U\pi_2\rangle{:}U(\alpha \times \beta) \rightarrowtail A \times B$$

is a regular mono. Thus, $\exists_p$ distributes over $\wedge$ and so

$$\begin{aligned}
\Box_{\alpha,\beta}(R \wedge S) &= \exists_p \Box_{\alpha\times\beta} p^* \\
&= \exists_p \Box_{\alpha\times\beta}(p^*R \wedge p^*S) \\
&= \exists_p(\Box_{\alpha\times\beta} p^*R \wedge \Box_{\alpha\times\beta} p^*S) && \text{(by Theorem 2.2.16)} \\
&= \exists_p \Box_{\alpha\times\beta} p^*R \wedge \exists_p \Box_{\alpha\times\beta} p^*S = \Box_{\alpha,\beta} R \wedge \Box_{\alpha,\beta} S.
\end{aligned}$$

$\square$

As we saw in Theorem 2.5.8, assuming the axiom of choice, bisimulations are relations which can be augmented with structure maps, making them relations in $\mathcal{E}_\Gamma$. The following corollary shows that, assuming Γ preserves pullbacks, the same result holds. Thus, under this (reasonably strong) assumption, the definition of bisimulation found in [**JR97**], etc., again coincides with our definition of bisimulation.

Corollary 2.5.27. *If U preserves regular relations, a relation R on A and B is a bisimulation iff there is a (necessarily unique) structure map $\rho{:}R \rightarrow \Gamma R$ making $\langle R, \rho\rangle$ a relation on $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$.*

Proof. Let $p = \langle U\pi_1, U\pi_2\rangle$ and let R be a bisimulation, $R = \exists_p \circ U_{\alpha\times\beta}\langle T, \tau\rangle$ for some $\langle T, \tau\rangle \in \mathsf{RegRel}(\alpha, \beta)$. Since p is a regular mono, $\exists_p T = T$ and so the result follows. □

2.5.3. The algebraic dual of bisimulations. A bisimulation is a relation between the carriers of two coalgebras which, loosely speaking, respects the structure maps of the coalgebras. In this way, a bisimulation is analogous to a pre-congruence. There is another structure on algebras which is related to the notion of a bisimulation — namely, the dual structure. For this, we explicitly dualize Definition 2.5.1 (regular relation).

Definition 2.5.28. Let A, B be objects in a category $\mathcal{C}$ with finite coproducts. A regular epi

$$p{:}A + B \longrightarrow\!\!\triangleright C$$

is called a *regular co-relation* on A and B.

Remark 2.5.29. A more general definition of regular co-relation can be found in [**BW85**], where one does not assume the category $\mathcal{C}$ has finite coproducts. See also Remark 2.5.2. We will not use this approach, but instead assume conditions sufficient to ensure that our category of algebras has coproducts.

Throughout the remainder, we assume that $\mathcal{E}$ is complete, exact, regularly co-well-powered and finitely cocomplete, that Γ preserves exact sequences and that the algebraic forgetful functor U is monadic. By Theorem 2.4.8, then, $\mathcal{E}^\Gamma$ has coproducts and by Theorem 1.4.11, $\mathcal{E}^\Gamma$ is exact. While we do not require exactness to dualize the preceding development of bisimulations, it does allow the dual theorems to be stated in familiar terms (i.e., in terms of equivalence relations instead of regular co-relations).

Let $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ be Γ-coalgebras. Let $\mathsf{RegCoRel}(A, B)$ be the category of regular corelations over A and B (in $\mathcal{E}$), and $\mathsf{RegCoRel}(\alpha, \beta)$ the category of regular corelations over $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$. Then, there is a forgetful functor

$$U_{\alpha,\beta}{:}\mathsf{RegCoRel}(\alpha, \beta) \longrightarrow \mathsf{RegCoRel}(A, B)$$

which takes a regular co-relation

$$p\colon \alpha + \beta \longrightarrow \langle C,\, \gamma\rangle$$

to the regular epi-mono factorization of $Up \circ [U\kappa_\alpha, U\kappa_\beta]$ (where κ_α and κ_β are the co-projections of the coproduct). See Figure 21.

$$\begin{tikzcd}
A + B \arrow[r, "{[U\kappa_\alpha, U\kappa_\beta]}"] \arrow[d] & U(\alpha + \beta) \arrow[d, "Up"] \\
U_{\alpha,\beta}\langle C,\, \gamma\rangle \arrow[r, tail] & C
\end{tikzcd}$$

FIGURE 21. The definition of $U_{\alpha,\beta}\colon \mathsf{RegCoRel}(\alpha, \beta) \to \mathsf{RegCoRel}(A, B)$.

Because both $\mathcal{E}$ and $\mathcal{E}^\Gamma$ are exact, we have isomorphisms

$$\mathsf{RegCoRel}(A, B) \cong \mathsf{EqRel}(A + B),$$
$$\mathsf{RegCoRel}(\alpha, \beta) \cong \mathsf{Cong}(\alpha + \beta).$$

We state the effect of $U_{\alpha,\beta}$ in terms of congruences on $\alpha+\beta$ and equivalence relations on $A + B$. Let $\langle R,\, \rho\rangle$ be a congruence on $\alpha + \beta$. Then $U_{\alpha,\beta}\langle R,\, \rho\rangle$ is given by the pullback of R along $[U\kappa_\alpha, U\kappa_\beta]$. In other words, elements x and y of $A+B$ are related by $U_{\alpha,\beta}\langle R,\, \rho\rangle$ if and only iff x and y are related by R as elements of $\alpha + \beta$.

2.6. Coinduction and bisimulations

The principle of coinduction from Section 1.5.3 is often expressed in terms of bisimulations. We follow that tradition in this section by restating the results of Theorem 1.5.25 in terms of bisimulations. To begin, we define what it means for elements of two coalgebras to be bisimilar. Then, we prove the usual statement of coinduction, namely, any two bisimilar elements of the final coalgebra are equal. The material found here differs from the standard presentation (say, in [**JR97**]) inasmuch as the definition of bisimulation (and, hence, bisimilar) differ from the standard definitions. As before, if $\mathcal{E}$ satisfies the axiom of choice, the definitions agree.

DEFINITION 2.6.1. Let $\langle A,\, \alpha\rangle$ and $\langle B,\, \beta\rangle$ be Γ-coalgebras and let $\langle a,\, b\rangle \in A \times B$. We say that a and b are *bisimilar*, denoted $a \sim_{\alpha,\beta} b$ or just $a \sim b$, if

$$\langle a,\, b\rangle \in \Box_{\alpha,\beta}(A \times B).$$

(Note that $\Box_{\alpha,\beta}(A \times B)$ is just $U_{\alpha,\beta}(\alpha \times \beta)$.)

Two elements are bisimilar just in case there is a bisimulation relating them, as the following theorem shows.

THEOREM 2.6.2. *$a \sim b$ iff there is a bisimulation R such that $\langle a, b\rangle \in R$. I.e., $a \sim b$ iff there is a coalgebraic relation $\langle R, \rho\rangle$ on $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ such that $\langle a, b\rangle \in U_{\alpha,\beta}\langle R, \rho\rangle$.*

PROOF. If $a \sim b$ then $\langle a, b\rangle$ is an element of the bisimulation $\Box_{\alpha,\beta}(A \times B)$. On the other hand, if $\langle a, b\rangle \in R$, where R is a bisimulation, then

$$\langle a, b\rangle \in R = \Box_{\alpha,\beta} R \leq \Box_{\alpha,\beta} A \times B.$$

$\Box$

Recall from Section 1.5.3 that a coalgebra is simple if it has no proper quotients.

THEOREM 2.6.3. *If $\langle A, \alpha\rangle$ is simple, then $\Box_{\alpha,\alpha}(A \times A) = \Delta_A$.*

PROOF. By Theorem 1.5.25, if $\langle A, \alpha\rangle$ is simple, then Δ_α is the largest relation on $\langle A, \alpha\rangle$. Hence, $[(]_{\alpha,\alpha} A \times A) = \Delta_\alpha$ and so (by Theorem 2.5.5)

$$\Box_{\alpha,\alpha}(A \times A) = U_{\alpha,\alpha}\Delta_\alpha = \Delta_A$$

$\Box$

COROLLARY 2.6.4. *If $\langle A, \alpha\rangle$ is simple then, for every element $\langle a, a'\rangle$ of $A \times A$, $a \sim a'$ iff $a = a'$.*

Theorem 2.6.2 and Corollary 2.6.4 provide the proof principle of coinduction: To prove two elements of a simple coalgebra are equal, it suffices to show that there is a bisimulation relating them.

The notion of bisimilarity is intended to capture the informal notion of *observational indistinguishability* (see [**JR97**] for another presentation of this viewpoint). A bisimulation is a relation that is preserved by applications of the structure maps. Think of the structure map for a coalgebra as a number of destructor operations that allow one to take a data structure apart and look at the substructures. For instance, the structure map for an $A \mapsto Z \times A$ coalgebra consists of two destructor functions: a head function, h_α, that gives the head of a stream, and a tail function, t_α, which returns the rest of the stream. We treat the elements of A as the internal state of the coalgebra, and so view them as unobservable, while the elements of Z are viewed as observable output. Hence, these destructors give a means of observing the behavior of the coalgebra A, by applying t_α some number of times, followed by h_α. This intuition regarding observable behavior can be made explicit for polynomial functors and similar inductively given classes of functors, but we do not do so here. See [**Jac99**] for an idea of how this is done, and see [**Cîr00**] for a more formal (and sophisticated) notion of a coalgebra observer.

With this informal notion of observations of a coalgebra, two elements of a coalgebra are bisimilar just in case they "look the same." The principle of coinduction, in this perspective, says that two elements of a simple coalgebra that look the same *are* the same. In order to justify this informal interpretation of coinduction, we will look at a few examples.

REMARK 2.6.5. The examples below involve coalgebras over **Set**. Consequently, we make use of the fact that, thanks to the axiom of choice, a relation R on A and B is a bisimulation iff there is a structure map $\rho : R \to \Gamma R$ such that $\langle R, \rho\rangle$ is a relation on $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$. See Theorem 2.5.8.

EXAMPLE 2.6.6. Consider the functor $\Gamma A = Z \times A$ above (see also Example 1.1.7). Let $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ be Γ-coalgebras, and $a \in A$ and $b \in B$. Then a is bisimilar to b just in case

$$h_\alpha(a) = h_\beta(b), \tag{13}$$

$$t_\alpha(a) \sim t_\beta(b). \tag{14}$$

Indeed, to prove that a and b satisfying (13) and (14) are bisimilar, we define a relation R on A and B by

$$cRd \leftrightarrow \exists n \,.\, t_\alpha^n(a) = c \wedge t_\beta^n(b) = d.$$

Then, it is easy to confirm that R is a bisimulation.

Bisimilarity for the functor $\Gamma A = Z \times A + 1$ is very similar. Let $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ be Γ-coalgebras (see Example 1.1.8). We can show that $a \sim b$ iff $\alpha(a) = *$ and $\beta(b) = *$ or if $\alpha(a) \neq *$, $\beta(b) \neq *$ and a, b satisfy (13) and (14).

EXAMPLE 2.6.7. Let $\mathbb{P}$ be a polynomial functor, and $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ be two $\mathbb{P}$-coalgebras (see Example 1.1.9). Then an element $a \in A$ is bisimilar to an element $b \in B$ just in case

$$\mathsf{label}(a) = \mathsf{label}(b), \tag{15}$$

$$\mathsf{br}(a) = \mathsf{br}(b), \tag{16}$$

$$\mathsf{child}_j(a) \sim \mathsf{child}_j(b) \text{ for all } j < \mathsf{br}(a). \tag{17}$$

EXAMPLE 2.6.8. Let **AtProp** be a collection of atomic propositions and consider the functor

$$\Gamma A = \mathcal{P}(\mathbf{AtProp}) \times \mathcal{P}(A).$$

A Γ-coalgebra is a Kripke model for the language $\mathcal{L}(\mathbf{AtProp})$ (see Example 1.1.10). Let $\mathfrak{A} = \langle A, \alpha\rangle$ and $\mathfrak{B} = \langle B, \beta\rangle$ be two such coalgebras, and $a \in A$, $b \in B$. We have

$a \sim b$ iff

$$\forall a' \in \pi_2 \circ \alpha(a)\, \exists b' \in \pi_2 \circ \beta(b)\,.\, a' \sim b',$$
$$\forall b' \in \pi_2 \circ \beta(b)\, \exists a' \in \pi_2 \circ \alpha(a)\,.\, a' \sim b',$$
$$\pi_2 \circ \alpha(a) = \pi_2 \circ \beta(b).$$

One can confirm, using these conditions, $a \sim b$ iff, for all $\phi \in \mathcal{L}(\mathbf{AtProp})$,

$$a \models_{\mathfrak{A}} \phi \text{ iff } b \models_{\mathfrak{B}} \phi.$$

See [**BM96**, Theorem 11.7] for the proof of this.

Example 2.6.9. Recall from Example 1.1.11 that coalgebras for the functor

$$\Gamma S = (\mathcal{P}_{\mathsf{fin}} S)^{\mathcal{I}}$$

can be viewed as automata taking input from $\mathcal{I}$. These are rather basic automata here, simply moving from one state to another, without giving any "output", and so the notion of bisimilarity is trivial. Namely, given any two coalgebras $\langle A,\, \alpha\rangle$and $\langle B,\, \beta\rangle$and any $a \in A$, $b \in B$, we have $a \sim b$.

To dress these automata up a bit, we will add a set of outputs, $\mathcal{O}$, and add a map taking each state to its output. In other words, we wish to consider coalgebras for the functor

$$\Delta S = \mathcal{O} \times (\mathcal{P}_{\mathsf{fin}} S)^{\mathcal{I}}.$$

One can show that, given $a \in U\langle A,\, \langle \alpha_o,\, \alpha_s\rangle\rangle$ and $b \in U\langle B,\, \langle \beta_o,\, \beta_s\rangle\rangle$ that $a \sim b$ just in case

- $\alpha_o(a) = \beta_o(b)$;
- for all i in $\mathcal{I}$ and all a' such that $a \xrightarrow{i} a'$, there is a b' such that $b \xrightarrow{i} b'$;
- for all i in $\mathcal{I}$ and all b' such that $b \xrightarrow{i} b'$, there is an a' such that $a \xrightarrow{i} a'$.

We will discuss the relationship between bisimulations and maps into the final coalgebra in more detail in Section 3.9. For now, we state a simple fact: bisimilar elements are mapped to the same element of the final coalgebra.

Theorem 2.6.10. *Let $\langle A,\, \alpha\rangle$ and $\langle B,\, \beta\rangle$ be Γ coalgebras, and $a \in A$, $b \in B$ be global points. Let*

$$!_\alpha : \langle A,\, \alpha\rangle \longrightarrow H1,$$
$$!_\beta : \langle B,\, \beta\rangle \longrightarrow H1$$

be the coalgebra homomorphisms into the final coalgebra. If the terminal object 1 *in $\mathcal{E}$ is projective with respect to epis, then*

$$a \sim b \textit{ implies } !_\alpha(a) = !_\beta(b).$$

If Γ preserves weak pullbacks, then the converse also holds.

PROOF. Let $a \sim b$. Then $\langle a, b\rangle \in U_{\alpha,\beta}(\alpha \times \beta)$, shown below.

$$\begin{tikzcd}
 & 1 \arrow[r, "{\langle a, b\rangle}"] \arrow[d, dotted, "c"'] & U_{\alpha,\beta}(\alpha \times \beta) \arrow[d, tail] \\
 & U(\alpha \times \beta) \arrow[ur, two heads, "p"] \arrow[r] & A \times B
\end{tikzcd}$$

Because 1 is projective with respect to epis, there is an element $c \in U(\alpha \times \beta)$ such that $p(c) = \langle a, b\rangle$. Now,

$$\begin{aligned}
!_\alpha(a) &= !_\alpha \circ \pi_A(\langle a, b\rangle) \\
&= !_\alpha \circ \pi_\alpha(c) \\
&= !_\beta \circ \pi_\beta(c) = !_\beta(b).
\end{aligned}$$

Under the assumption that Γ preserves weak pullbacks, then so does U [**JPT$^+$98**, Lemma 2.8]. Thus, the diagram below is a weak pullback.

$$\begin{tikzcd}
U(\alpha \times \beta) \arrow[r] \arrow[d] & A \arrow[d, "!_\alpha"] \\
B \arrow[r, "!_\beta"'] & UH1
\end{tikzcd}$$

Hence, if $!_\alpha \circ \pi_A(\langle a, b\rangle) = !_\beta \circ \pi_B(\langle a, b\rangle)$, then $\langle a, b\rangle$ factors through $U(\alpha \times \beta)$ and so $a \sim b$. $\square$

2.7. n-simulations

One can generalize bisimulations to include n-simulations. This allows a more uniform treatment of these distinguished relations in an internal logic in Section 4.1.2. We briefly present the definitions and main theorems here.

A *regular nary relation@regular n-ary relation* over $A_1, \ldots, A_n$ is a regular subobject of $\prod A_i$.

For each finite family $\langle A_1, \alpha_1\rangle, \ldots, \langle A_n, \alpha_n\rangle$ of coalgebras, we define a map

$$U_{\alpha_1,\ldots,\alpha_n} : \mathsf{SubCoalg}(\textstyle\prod \alpha_i) \longrightarrow \mathsf{RegSub}(\textstyle\prod A_i),$$

by $U_{\alpha_1,\ldots,\alpha_n} = \exists_{\langle U\pi_1,\ldots,U\pi_n\rangle} \circ U_{\prod \alpha_i}$. We define the category $n\text{-}\mathsf{sim}(\alpha_1, \ldots, \alpha_n)$ to be the image of this functor.

The functor $U_{\alpha_1,\ldots,\alpha_n}$ has a right adjoint, $[-]_{\alpha_1,\ldots,\alpha_n}$, defined by

$$[-]_{\alpha_1,\ldots,\alpha_n} = [-]_{\prod \alpha_i} \circ \langle U\pi_1, \ldots, U\pi_n\rangle^*.$$

This gives rise to a comonad $\Box_{\alpha_1,\ldots,\alpha_n}$ on $\mathsf{RegSub}(\prod A_i)$, and a monad $\bigtriangledown_{\alpha_1,\ldots,\alpha_n}$ on $\mathsf{SubCoalg}(\prod \alpha_i)$, that is, an interior operator and a closure operator, respectively. The $\Box$ operator takes a relation to the largest n-simulation contained in it, while the

∇ operator takes a coalgebraic relation to the largest relation with the same image (under $U_{\alpha_1,\dots,\alpha_n}$).

In Section 2.5.2, we showed that the bisimulation modal operator is normal if the endofunctor Γ preserves regular relations. The following theorems shows that the same assumption suffices to conclude that the n-simulation modal operate is also normal (for any n).

THEOREM 2.7.1. *If Γ preserves regular binary relations and regular monos, then Γ preserves regular n-ary relations.*

PROOF. By induction on n. The case for $n = 1, 2$ is by assumption. Suppose that Γ preserves regular n-ary relations. It suffices to show that, given a family $\{A_i\}_{i<n+1}$, $\Gamma \prod A_i$ is a regular subobject of $\prod \Gamma A_i$. By inductive hypothesis, $\Gamma \prod_{i<n} A_i$ is a regular subobject of $\prod_{i<n} \Gamma A_i$. Hence, we have

$$\Gamma(\textstyle\prod_{i<n} A_i \times A_n) \rightarrowtriangle \Gamma \prod_{i<n} A_i \times \Gamma A_n \rightarrowtriangle \prod_{i<n} \Gamma A_i \times \Gamma A_n \ ,$$

completing the proof. $\square$

THEOREM 2.7.2. *If $\Gamma : \mathcal{E} \to \mathcal{E}$ preserves regular relations and pullbacks along regular monos, then, for any finite family*

$$\langle A_1, \alpha_1\rangle, \dots, \langle A_n, \alpha_n\rangle,$$

$\Box_{\alpha_1,\dots,\alpha_n}$ is a normal necessity operator (although it need not preserve $\top$ and so is typically not **S4***).*

PROOF. As before, it suffices to show that $\Box$ distributes over $\wedge$. One uses the fact that U preserves regular n-ary relations and thus $\langle U\pi_1, \dots, U\pi_n\rangle$ is a regular mono. Hence, $\exists_{\langle U\pi_1,\dots,U\pi_n\rangle}$ distributes over $\wedge$. By assumption, U preserves pullbacks along regular monos, and hence, intersections, and so, since

$$\Box_{\alpha_1,\dots,\alpha_n} = \exists_{\langle U\pi_1,\dots,U\pi_n\rangle} U_{\prod \alpha_i}[-]_{\prod \alpha_i} \langle U\pi_1, \dots, U\pi_n\rangle^*,$$

the result follows. $\square$

The following theorem and corollary are obvious generalizations of Theorem 2.5.19 and Corollary 2.5.20. We omit the proof of these theorems, and prove a related theorem and corollary hereafter (Theorem 2.7.5 and its corollary).

THEOREM 2.7.3. *Let $\{f_i : \langle A_i, \alpha_i\rangle \to \langle B_i, \beta_i\rangle\}_{1 \le i \le n}$ be homomorphisms. Then*

$$[-]_{\alpha_1,\dots,\alpha_n} \circ \left(\prod U f_i\right)^* = \left(\prod f_i\right)^* [-]_{\beta_1,\dots,\beta_n}.$$

COROLLARY 2.7.4. *Under the same conditions as Theorem 2.7.3,*

$$U_{\beta_1,\dots,\beta_n} \circ \exists_{\prod f_i} = \exists_{\prod U f_i} \circ U_{\alpha_1,\dots,\alpha_n}.$$

The same facts hold when we replace the products of maps in Theorem 2.7.3 and Corollary 2.7.4 with projections.

THEOREM 2.7.5. *Let $\langle A_1, \alpha_1\rangle, \ldots, \langle A_n, \alpha_n\rangle$ be given. Then*

$$[-]_{\alpha_1,\ldots,\alpha_n} \circ {\pi_i}^* = {\pi_i}^* \circ [-]_{\alpha_i}$$

(where π_i on the left hand side is the projection in $\mathcal{E}$, while on the right hand side, it is the projection in $\mathcal{E}_\Gamma$.

PROOF. In Figure 22, the left hand face is a pullback by (the generalization of) Theorem 2.5.12, and the rear face is a pullback because H preserves pullbacks. The right hand face is a pullback by Theorem 2.2.11. To confirm that the front face is a pullback, and hence $[\pi_i^* R] = \pi_i^*[R]$, it suffices to show that the bottom face commutes.

$$\begin{tikzcd}
H\pi_i^*R \arrow[rr] \arrow[dd] & & HR \arrow[dd, tail] & \\
& {[\pi_i^*R]} \arrow[ul] \arrow[rr] \arrow[dd, tail] & & {[R]} \arrow[ul] \arrow[dd, tail] \\
H\prod A_i \arrow[rr] & & HA & \\
& \prod \alpha_i \arrow[ul] \arrow[rr] & & \langle A_i, \alpha_i \rangle \arrow[ul]
\end{tikzcd}$$

FIGURE 22. $[-]$ commutes with pullback along a projection.

The map $\langle A, \alpha\rangle \to HA$ is the adjoint transpose of the identity, i.e., η_α, and the map $\prod \alpha_i \to H \prod A_i$ is (up to the isomorphism $H\prod A_i \cong \prod HA_i$) the unit $\prod \eta_{\alpha_i}$. Thus, we see that the bottom face commutes by naturality. □

COROLLARY 2.7.6. *$U_{\alpha_1,\ldots,\alpha_n}$ commutes with $\exists_{\pi_i}$.*

REMARK 2.7.7. Theorem 2.7.5 and its corollary also apply when we replace π_i with a tuple of projections

$$\langle \pi_{i_1}, \ldots, \pi_{i_n}\rangle : \prod \alpha_i \longrightarrow \prod \alpha_{i_j}\,.$$

Since, in this case, the $[-]$ operator for the image is the n-simulation operator (and not merely the subcoalgebra operator), this is a non-trivial observation.

This last theorem will be useful in Chapter 4, where we introduce an internal version of the $\Box$ operators from this chapter. The theorem will be used in Theorem 4.2.3 to yield an axiom for $\Box$ in the internal logic.

THEOREM 2.7.8. *Let $\langle A_i, \alpha_i\rangle$, $\langle B, \beta\rangle$ be given, and $\pi_B : B \times \prod A_i \to \prod A_i$ be the evident projection (here, we've subscripted the projection with the object that we're projecting out). Then $\Box \pi_B{}^* \leq \pi_B{}^* \Box$.*

PROOF. Let $\pi_\beta : \beta \times \prod \alpha_i \to \prod \alpha_i$ be the corresponding projection in $\mathcal{E}_\Gamma$, and let

$$\begin{aligned} &p : U(\beta \times \textstyle\prod \alpha_i) \longrightarrow B \times \textstyle\prod A_i\,, \\ &q : U \textstyle\prod \alpha_i \longrightarrow \textstyle\prod A_i \end{aligned}$$

be the evident maps, so that $\pi_B \circ p = q \circ U\pi_\beta$. We omit the subscripts for $\Box$, U and $[-]$ in the following calculation, but these should be clear from context.

$$\begin{aligned} \exists_{\pi_B} \circ \Box \circ \pi_B{}^* &= \exists_{\pi_B} \circ \exists_p \circ \Box \circ p^* \circ \pi_B{}^* \\ &= \exists_q \circ \exists_{U\pi_\beta} \circ \Box \circ (U\pi_\beta)^* \circ q^* \\ &= \exists_q \circ \exists_{U\pi_\beta} \circ U \circ \pi_\beta{}^* \circ [-] \circ q^* && \text{(by Corollary 2.2.8)} \\ &= \exists_q \circ U \circ \exists_{\pi_\beta} \circ \pi_\beta{}^* \circ [-] \circ q^* && \text{(by Theorem 2.2.5)} \\ &\leq \exists_q \circ U \circ [-] \circ q^* = \Box. \end{aligned}$$

Hence, by the adjunction $\exists_{\pi_B} \dashv \pi_B{}^*$, the result follows. $\Box$

CHAPTER 3

Birkhoff's variety theorem

In this chapter, we give an extended example of the categorical approach to classical theorems in universal algebra. The Birkhoff variety theorem [**Bir35**] relates closure conditions on classes of universal algebras (for a fixed signature) to defining equations for the class. We begin by stating the classical theorem. Following this, we translate the relevant ideas to the categorical setting that has been developed in the preceding chapters.

We give two versions of the variety theorem: In the first, we ignore the features of categories of algebras and prove an abstract theorem that applies to many categories. This abstract theorem doesn't discuss equational definability explicitly, since a category requires a certain amount of structure before the notion of equations makes sense. Instead, we state the abstract version of the variety theorem strictly in terms of orthogonality conditions.

We can then apply the abstract theorem to categories of algebras, where we do have a suitable notion of equation (assuming that the algebraic forgetful functor is monadic). This allows us to recover the classical theorem, assuming the traditional setting. We conclude our discussion of equations in categories of algebras with a presentation of Birkhoff's deductive completeness theorem in terms of closure operators on equations over X.

Following this, we dualize the previous work to prove, first, an abstract covariety theorem, and then a covariety theorem for categories of coalgebras. Because the variety theorem was proved for categories of algebras over an abstract category, the real work for the covariety theorem has already been done — although one must still confirm that the dual setting (a co-Birkhoff category) is a reasonable setting. One still must interpret the terms of the dualized theorem, which yields definitions of *coequations* and *covariety*. The strengthening of the variety theorem to the classical result (where each variety is equationally definable over a single set of variables — see Section 3.4 on "uniformly Birkhoff categories") does not directly dualize, however. Some work is required to capture the similar result for categories of coalgebras.

Following our presentation of the covariety theorems, we present the dual of the deductive completeness theorem, which states that a coequation φ is the minimal coequation satisfied by some class of coalgebras just in case φ is an *endomorphism-invariant* subcoalgebra. We conclude the chapter with a discussion of a distinguished class of covarieties, the *behavioral* covarieties. These covarieties were first studied in [**GS98**], where they were called "complete covarieties". We present a similar account, while relating Gumm and Schröder's work to Grigore Roşu in [**Roş01**], where the same class of covarieties are called "sinks".

3.1. The classical theorem

We fix a signature Σ and consider classes of Σ-algebras. Birkhoff's variety theorem says that a class $\mathbf{V}$ of Σ-algebras is closed under products, subalgebras and homomorphic quotients just in case it is equationally definable. In this section, we define these terms and state the theorem.

In order to present the theorem in its historical form, we will use the language of universal algebras (algebras for a signature of function symbols). In Section 3.2, we will restate the definitions in the terms of Γ-algebras and explore the role of equations in greater detail. Accordingly, in this section, we state the definitions and theorems in the notation of Example 1.1.5 (Σ-algebras). Thus, recall that a Σ-algebra is a pair,

$$\mathcal{S} = \langle S, \{ f_{\mathcal{S}}^{(n)} : S^n \longrightarrow S \,|\, f^{(n)} \in \Sigma \} \rangle,$$

consisting of a set S together with interpretations for the function symbols of Σ. A *subalgebra* of $\mathcal{S}$ is a Σ-algebra,

$$\mathcal{T} = \langle T, \{ f_{\mathcal{T}}^{(n)} : T^n \longrightarrow T \,|\, f^{(n)} \in \Sigma \} \rangle,$$

such that $T \subseteq S$ and each $f_{\mathcal{T}}^{(n)}$ is the restriction of $f_{\mathcal{S}}^{(n)}$ to T. If

$$\mathcal{S}_i = \langle S_i, \{ f_{\mathcal{S}_i}^{(n)} : S_i^n \longrightarrow S_i \,|\, f^{(n)} \in \Sigma \} \rangle$$

is a family of Σ-algebras, then the product $\prod \mathcal{S}_i$ exists and has as carrier $\prod S_i$. The interpretation of f^n on $\prod S_i$ is given by

$$\prod f_{\mathcal{S}_i}^{(n)} : (\prod S_i)^n \longrightarrow \prod S_i$$

(via the isomorphism $(\prod S_i)^n \cong \prod (S_i)^n$). In other words, the interpretation is given component-wise.

Definition 3.1.1. Let $\mathbf{V}$ be a class of Σ-algebras. We say that $\mathbf{V}$ *is closed under subalgebras* if, whenever $\mathcal{S}$ is in $\mathbf{V}$ and $\mathcal{T}$ is a subalgebra of $\mathcal{S}$, then $\mathcal{T}$ is in $\mathbf{V}$. If, whenever each $\mathcal{S}_i$ is in $\mathbf{V}$, then $\prod \mathcal{S}_i$ is in $\mathbf{V}$, we say that $\mathbf{V}$ *is closed under products*. We say that $\mathbf{V}$ *is closed under quotients* if, whenever a homomorphism

$$p : \mathcal{S} \longrightarrow\!\!\!\!\triangleright\, \mathcal{T}$$

is a regular epi and $\mathcal{S} \in \mathbf{V}$, then $\mathcal{T} \in \mathbf{V}$.

DEFINITION 3.1.2. Let $\mathbf{V}$ be a class of Σ-algebras. If $\mathbf{V}$ is closed under subalgebras, products and homomorphic quotients, then $\mathbf{V}$ is called a *Birkhoff variety*.

We now turn to equational definability. We use the fact that Σ-algebras have free algebras in order to define an equation. Given a set (of variables) X, the free algebra over X (denoted FX with carrier UFX) is the collection of Σ-terms over the variables in X (Section 2.1.2). Thus, we can view an equation $\tau_1 = \tau_2$ over X as a pair of elements of UFX.

Let

$$\mathcal{S} = \langle S, \{f^{(n)}_{\mathcal{S}_i} : S_i^n \longrightarrow S_i \,|\, f^{(n)} \in \Sigma\}\rangle$$

be a Σ-algebra. The property of freeness states that, for every assignment σ of the variables of X to S (i.e., for every $\mathbf{Set}$ map $\sigma : X \to S$), there is a unique homomorphic extension

$$\widetilde{\sigma} : FX \longrightarrow \mathcal{S}.$$

An algebra $\mathcal{S}$ satisfies the equation $\tau_1 = \tau_2$ (denoted $\mathcal{S} \models \tau_1 = \tau_2$) just in case, under every such assignment σ, we have

$$\widetilde{\sigma} \circ \tau_1 = \widetilde{\sigma} \circ \tau_2.$$

Given a set E of equations over X, we write

$$\mathcal{S} \models E$$

just in case $S \models \tau_1 = \tau_2$ for every equation $\tau_1 = \tau_2$ in E. We define

$$\mathsf{Mod}(E) = \{\mathcal{S} \mid \mathcal{S} \models E\}.$$

The set notation in this definition should not be taken literally. In general, $\mathsf{Mod}(E)$ is a proper class.

DEFINITION 3.1.3. Let $\mathbf{V}$ be a class of Σ-algebras. We say that $\mathbf{V}$ is an *equational variety* just in case there is a set of variables X and a set E of equations over X such that

$$\mathbf{V} = \mathsf{Mod}(E).$$

THEOREM (Birkhoff's variety theorem). *Let* $\mathbf{V}$ *be a class of* Σ*-algebras. Then* $\mathbf{V}$ *is a Birkhoff variety iff* $\mathbf{V}$ *is an equational variety.*

3.2. A categorical approach

We now translate Birkhoff's variety theorem to categorical terms. As we've seen, the category of algebras for a signature, $\mathsf{Alg}(\Sigma)$ is isomorphic to the category $\mathbf{Set}^{\mathbb{P}}$ for a related polynomial functor $\mathbb{P}$ (see Example 1.1.5). In this section, we translate the remaining terms of Section 3.1 into categorical terms and prove an abstract version of the variety theorem, which holds in a wide variety of categories (and not just categories of algebras).

3.2.1. Birkhoff categories. We begin by describing some of the properties of $\mathsf{Alg}(\Sigma)$ that are relevant to Birkhoff's theorem. In particular, we want to pay close attention to those properties that lead to natural definitions of Birkhoff variety and equational variety in abstract categories. We will call any category which has the requisite structure a *Birkhoff category.* We can then prove an abstract version of the variety theorem. It is just a little work to show that, for a wide variety of base categories and a wide variety of functors, the category $\mathcal{E}^{\Gamma}$ is a Birkhoff category. In particular, we will show that, for polynomials $\mathbb{P}$, the category $\mathbf{Set}^{\mathbb{P}}$ (and hence $\mathsf{Alg}(\Sigma)$) is a Birkhoff category, and so the abstract Birkhoff theorem applies. This does not immediately lead to the classical theorem, however. Rather, the direct consequence of the abstract variety for categories of algebras is that every variety is defined by a class (not a set) of equations. In order to show that a set of equations suffices, we need to show that $\mathbf{Set}^{\mathbb{P}}$ is *uniformly Birkhoff* (see Section 3.4).

Recall that a category is regularly co-well-powered just in case each object has only set-many quotients (Definition A.3.1). We say that an object A is regular projective if it is projective with respect to regular epis, so that, for every regular epi $B \twoheadrightarrow C$ and map $A \to C$, there is a (not necessarily unique) map $A \to B$ making the diagram below commute.

$$\begin{tikzcd} A \arrow[r, dotted] \arrow[dr] & B \arrow[d, twoheadrightarrow] \\ & C \end{tikzcd}$$

A category *has enough regular projectives* just in case every object is a quotient of some regular projective.

DEFINITION 3.2.1. A *quasi-Birkhoff category* is a category that is regularly co-well-powered, complete and has regular epi-mono factorizations. A *Birkhoff category* is a quasi-Birkhoff category with enough regular projectives.

The Birkhoff categories have the structure necessary for a notion of Birkhoff variety. We postpone the generalization of equational variety until we examine equational definability in $\mathcal{E}^{\Gamma}$ in more detail.

DEFINITION 3.2.2. Let $\mathcal{C}$ be a quasi-Birkhoff category and $\mathbf{V}$ a full subcategory of $\mathcal{C}$. Then $\mathbf{V}$ is a *quasi-Birkhoff variety* (or just *quasi-variety*)iff $\mathbf{V}$ is closed under products and subobjects. $\mathbf{V}$ is a *Birkhoff variety* if $\mathcal{C}$ is a Birkhoff category and $\mathbf{V}$ is a quasi-Birkhoff variety closed under quotients (codomains of regular epis).

REMARK 3.2.3. Any quasi-variety is closed under isomorphisms, since it is closed under subobjects.

One may define these closure conditions in terms of fixed points for operators on subcategories of $\mathcal{C}$. One defines the operator $H : \mathsf{Sub}(\mathcal{C}) \to \mathsf{Sub}(\mathcal{C})$ to take a class $\mathbf{V}$ to

$$H\mathbf{V} = \mathbf{V} \cup \{C \in \mathcal{C} \mid \exists K \in \mathbf{V} \exists q : K \twoheadrightarrow C\}$$

(abusing set notation here). Similarly, one defines operators S and P taking $\mathbf{V}$ to the classes

$$S\mathbf{V} = \mathbf{V} \cup \{C \in \mathcal{C} \mid \exists K \in \mathbf{V} \exists q : K \rightarrowtail C\},$$
$$P\mathbf{V} = \{\prod_{i \in I} C_i \mid C_i \in \mathcal{C},\ I \in \mathbf{Set}\}.$$

Then $\mathbf{V}$ is a quasi-Birkhoff variety iff $\mathbf{V} = SP\mathbf{V}$ and a variety just in case $\mathbf{V} = HSP\mathbf{V}$. We don't make use of these operators hereafter, but see [**GS98**] for a presentation along these lines.

3.2.2. Equations in $\mathcal{E}^{\Gamma}$. In Section 3.1, we discussed equations for universal algebras. We now use that work to give an account of equations for Γ-algebras generally. Our goal is to find a categorical property that generalizes the notion of equational definability to a wider class of categories — including categories which are *not* monadic over some base category. As we will see in Section 3.2.5, equational definability is generalized by orthogonality to a regular epi with regular projective domain.

In order to interpret equations over X in $\mathcal{E}^{\Gamma}$, we require that Γ is a varietor (i.e., the algebraic forgetful functor

$$U : \mathcal{E}^{\Gamma} \longrightarrow \mathcal{E}$$

is monadic). Also, for this section, we assume that $\mathcal{E}$ is a Birkhoff category that has all coequalizers. Thus, by Theorem 2.4.2, $\mathcal{E}^{\Gamma}$ has all coequalizers. This assumption isn't necessary for the final proof of Birkhoff's variety theorem, but is useful in understanding the role of equations in $\mathcal{E}^{\Gamma}$.

Let X be a set of variables. Then an equation over X is a pair of elements of UFX, written $\tau_1 = \tau_2$. Equivalently, an equation is a pair of maps

$$1 \underset{\tau_2}{\overset{\tau_1}{\rightrightarrows}} UFX\ .$$

Similarly, a set of equations E is given by a pair of jointly monic maps

$$E \underset{e_2}{\overset{e_1}{\rightrightarrows}} UFX\ .$$

Recall the definition of satisfaction from Section 3.1. A $\mathbb{P}$-algebra $\langle A, \alpha\rangle$ satisfies the equations in E just in case, for all $\sigma : X \to A$, the extension $\widetilde{\sigma} : FX \to \langle A, \alpha\rangle$ equalizes e_1 and e_2. That is,

$$\langle A, \alpha\rangle \models E \text{ iff for all } \sigma : X \longrightarrow A,\ U\widetilde{\sigma} \circ e_1 = U\widetilde{\sigma} \circ e_2.$$

Let $\widetilde{e}_1$ and $\widetilde{e}_2$ be the adjoint transposes of e_1 and e_2, respectively. Let

$$q_E : FX \longrightarrow \langle Q_E, \nu_E\rangle$$

be the coequalizer of $\widetilde{e}_1$ and $\widetilde{e}_2$, shown below[1].

$$\begin{tikzcd} FE \arrow[r, shift left, "\widetilde{e}_1"] \arrow[r, shift right, "\widetilde{e}_2"'] & FX \arrow[r, "q_E"] & \langle Q_E, \nu_E\rangle \end{tikzcd}$$

We note that $\widetilde{\sigma}$ equalizes $\widetilde{e}_1$ and $\widetilde{e}_2$ just in case $U\widetilde{\sigma}$ equalizes e_1 and e_2. Thus, $\langle A, \alpha\rangle \models E$ just in case, for every homomorphism

$$\widetilde{\sigma} : FX \longrightarrow \langle A, \alpha\rangle,$$

there is a unique homomorphism

$$\overline{\sigma} : \langle Q_E, \nu_E\rangle \longrightarrow \langle A, \alpha\rangle$$

such that the diagram below commutes.

$$\begin{tikzcd} FX \arrow[r, "U\widetilde{\sigma}"] \arrow[d, "q_E"'] & \langle A, \alpha\rangle \\ \langle Q_E, \nu_E\rangle \arrow[ur, "\overline{\sigma}"'] & \end{tikzcd}$$

We take this property as central to a generalization of equation satisfaction. We recall the definition of orthogonality, which can be found in [**Bor94**] and other introductory texts.

Definition 3.2.4. A map $f : A \to B$ is called *orthogonal to an object* X (written $f \perp X$) if, for every map $a : A \to X$, there is a unique map $b : B \to X$ such that $a = b \circ f$.

Thus, $\langle A, \alpha\rangle \models E$ iff $q_E \perp \langle A, \alpha\rangle$.

This leads to the following definition of equational variety:

Definition 3.2.5. Let $\mathcal{E}^{\Gamma}$ be a quasi-Birkhoff category and let $\mathbf{V}$ be a full subcategory of $\mathcal{E}^{\Gamma}$. We say that $\mathbf{V}$ is an *equational variety* if

$$\mathbf{V} = \{\langle A, \alpha\rangle \mid q \perp \langle A, \alpha\rangle\}$$

for some regular epi q with domain FX (for some $X \in \mathcal{E}$).

[1] We could instead consider the coequalizer of $\langle E\rangle$, the pre-congruence containing E (see Section 1.4.2) The coequalizer of $\langle E\rangle$ is isomorphic to the coequalizer of $\widetilde{e}_1$ and $\widetilde{e}_2$, though we omit the proof.

Equivalently, following the presentations of [**AN81a, BH76, AR94**], etc., one could say that an equational variety is just the *injectivity class* of some quotient $FX \twoheadrightarrow Q$. The author discovered these alternative approaches after developing the theory in terms of orthogonality, and we present that development here.

3.2.3. Orthogonality. Definition 3.2.5 indicates the basic approach that we take: orthogonality is a generalization of satisfaction of a set of equations. In this section, we introduce some notation for discussing orthogonality and state some basic results.

If S is a collection of arrows of $\mathcal{C}$, we write $S \perp C$ if $f \perp C$ for all $f \in S$. Similarly, if $\mathbf{V}$ is a collection of objects (equivalently, a full subcategory) of $\mathcal{C}$, we write $f \perp \mathbf{V}$ if $f \perp C$ for each $C \in \mathbf{V}$. We define the notation $S \perp \mathbf{V}$ in the obvious way.

Given a category $\mathcal{C}$ and a collection of maps S in $\mathcal{C}$, $S^\perp$ is the collection of all objects C of $\mathcal{C}$ such that $S \perp C$. Similarly, given a collection of objects $\mathbf{V}$ of $\mathcal{C}$, $\mathbf{V}^\perp$ is the collection of all arrows f in $\mathcal{C}$ such that $f \perp \mathbf{V}$.

In these terms, $\mathbf{V}$ is an equational variety just in case $\mathbf{V} = \{q : FX \twoheadrightarrow \bullet\}^\perp$ for some regular epi q.

The class of all collections of maps of $\mathcal{C}$ forms a poset, $\mathsf{Sub}(\mathcal{C}_1)$, taking inclusion as the ordering. Similarly, the class of all full subcategories of $\mathcal{C}$ forms a poset, $\mathsf{Sub}(\mathcal{C}_0)$. Thus, the $\perp$ operators are maps between posets. Since $S \subseteq T$ implies $S^\perp \supseteq T^\perp$, and likewise for the $\perp$ operator with domain $\mathsf{Sub}(\mathcal{C}_0)$, we can view these operators as functors

$$\mathsf{Sub}(\mathcal{C}_1) \rightleftarrows (\mathsf{Sub}(\mathcal{C}_0))^{\mathsf{op}}$$

It is easy to see that, given a collection of maps S and a full subcategory $\mathbf{V}$, $S^\perp \subseteq \mathbf{V}$ iff $S \supseteq \mathbf{V}^\perp$. Thus, the two $\perp$ functors form a Galois correspondence (see [**Bor94**, Volume 1, Example 3.1.6.m]) and so $\perp\perp$ is a closure operation.

Given a collection of arrows, S, we say that S *spans the collection of arrows* $S^{\perp\perp}$. In particular, if $S^\perp = \mathbf{V}$, then S spans $\mathbf{V}^\perp$. Because the $\perp$ functors form a Galois correspondence, $S^\perp = S^{\perp\perp\perp}$. Thus, if $S^\perp = \mathbf{V}$, we have $\mathbf{V} = \mathbf{V}^{\perp\perp}$. In this case, we say that $\mathbf{V}$ *is closed*.

Remark 3.2.6. The subcategory $S^\perp$ is denoted $\mathsf{Inj(S)}$ by some authors, to denote the collection of objects which are *injective with respect to* S.

3.2.4. An abstract version of Birkhoff's theorem. In this section, we prove a quasi-variety theorem for abstract categories. This theorem is essentially found in [**BH76**] and is generalized in various articles by Andreéka and Németi, but was independently proven by the author before being referred to these articles[2].

[2]Thanks to Jiří Adámek and an anonymous reviewer for [**Hug01**] for these references.

THEOREM 3.2.7. *Let $\mathcal{C}$ be a quasi-Birkhoff category and $\mathbf{V}$ a full subcategory of $\mathcal{C}$. The following are equivalent.*

(1) *$\mathbf{V}$ is closed under products and subobjects (i.e., $\mathbf{V}$ is a quasi-variety).*
(2) *$\mathbf{V}$ is a regular epi-reflective subcategory of $\mathcal{C}$. That is, a subcategory whose inclusion $U^{\mathbf{V}}:\mathbf{V}\to\mathcal{C}$ has a left adjoint $F^{\mathbf{V}}$ such that each component of the unit $\eta^{\mathbf{V}}:U^{\mathbf{V}}F^{\mathbf{V}}\to 1_{\mathcal{C}}$ is a regular epi.*
(3) *$\mathbf{V}$ is closed. I.e., $\mathbf{V} = S^{\perp}$ for some collection S of regular epis.*

PROOF. We prove each implication in turn.

(1)⇒(2): We first show that the inclusion $U^{\mathbf{V}}$ has a left adjoint. Since $\mathbf{V}$ is closed under limits, it suffices, by the adjoint functor theorem ([**Bor94**, Volume 1,Theorem 3.3.3]), to show that for each $C \in \mathcal{C}$, there is a set of objects $\Theta_C \subseteq \mathbf{V}$ such that for each $K \in \mathbf{V}$ and each $f:C\to K$ in $\mathcal{C}$, f factors through some $K' \in \Theta_C$.

Take Θ_C to be the collection of quotients of C in $\mathbf{V}$. This is a set, since $\mathcal{C}$ is regularly co-well-powered. Given any $f:C\to K$ with $K \in \mathbf{V}$, we take the regular epi factorization of f, shown below.

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, two heads] & K \\ K' \arrow[ur, tail] & \end{tikzcd}$$

Then K' is in $\mathbf{V}$, since $\mathbf{V}$ is closed under subobjects. Thus, we may take K' to be an object of Θ_C.

Because $\mathbf{V}$ is closed under subobjects, the reflection is a regular epireflection ([**Bor94**, Volume 1, Proposition 3.6.4]).

(2)⇒(3): We will show that $(\eta^{\mathbf{V}})^{\perp} = \mathbf{V}$. That $\mathbf{V} \subseteq (\eta^{\mathbf{V}})^{\perp}$ is obvious from the characteristic property of $\eta^{\mathbf{V}}:1\Rightarrow U^{\mathbf{V}}F^{\mathbf{V}}$. We will show the other inclusion. Accordingly, suppose that $\eta^{\mathbf{V}} \perp C$. Then $\eta^{\mathbf{V}}_C \perp C$ in particular and thus, there is a map $\overline{\mathsf{id}}:U^{\mathbf{V}}F^{\mathcal{C}}C\to C$ such that the diagram below commutes:

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d, two heads, "\eta^{\mathbf{V}}_C"'] & C \\ U^{\mathbf{V}}F^{\mathbf{V}}C \arrow[ur, "\overline{\mathsf{id}}"'] & \end{tikzcd}$$

Since $\eta^{\mathbf{V}}_C$ is thus both regular epi and mono, it is an isomorphism. Since $\mathbf{V}$ is closed under isomorphisms (Remark 3.2.3), $C \in \mathbf{V}$.

(3)⇒(1): Let S be a collection of regular epis and $\mathbf{V} = S^{\perp}$. It is easy to see that $\mathbf{V}$ is closed under products. Suppose that $K' \in \mathbf{V}$ and $i:K\rightarrowtail K'$. Let $f:A\twoheadrightarrow B \in S^{\perp}$ and $g:A\to K$ be given, as in Figure 1. Then, since $f \perp K'$, there is a unique map $\overline{\imath \circ g}:B\to K'$ such that $\overline{\imath \circ g} \circ f = i \circ g$. Since f is regular and hence strong, there is a unique map $\overline{g}$, as shown, making the diagram commute.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & K \arrow[d, "i"] \\ B \arrow[r, "\overline{i\circ g}"'] \arrow[ur, dotted, "\overline{g}"] & K' \end{tikzcd}$$

FIGURE 1. $S^{\perp}$ is closed under subobjects.

□

COROLLARY 3.2.8. *Let $\mathcal{C}$ be a quasi-Birkhoff category and $\mathbf{V}$ a quasi-variety of $\mathcal{C}$. Then*

(1) $\mathbf{V} = (\eta^{\mathbf{V}})^{\perp}$.
(2) For each $C \in \mathcal{C}$, $C \in \mathbf{V}$ iff $\eta^{\mathbf{V}}_C \perp C$, where $\eta^{\mathbf{V}}$ is the unit of the adjunction $F^{\mathbf{V}} \dashv U^{\mathbf{V}}$.
(3) The counit $\varepsilon^{\mathbf{V}} : F^{\mathbf{V}}U^{\mathbf{V}} \to 1_{\mathbf{V}}$ is an isomorphism.
(4) The corresponding monad, $T^{\mathbf{V}} = U^{\mathbf{V}}F^{\mathbf{V}}$, is idempotent.
(5) The monad $T^{\mathbf{V}}$ preserves regular epis.

PROOF. We sketch each item in turn.

(1) See the proof of (2)⇒(3) in Theorem 3.2.7.
(2) If $C \in \mathbf{V}$, then $C \perp \eta^{\mathbf{V}}_C$ by (1). On the other hand, if $C \perp \eta^{\mathbf{V}}_C$, then $C \cong T^{\mathbf{V}}C$ by the proof of (2)⇒(3) in Theorem 3.2.7.
(3) The functor $U^{\mathbf{V}}$ is full and faithful, so [**Bor94**, Proposition 3.4.1, Volume 1] applies.
(4) This follows from [**Bor94**, Volume 2, Theorem 4.2.4], and can also be seen directly in the proof of (2)⇒(3).
(5) Let $q : A \to Q$ be a regular epi. Since $T^{\mathbf{V}}q \circ \eta^{\mathbf{V}}_A = \eta^{\mathbf{V}}_Q \circ q$ and the right hand side is a regular epi, so is $T^{\mathbf{V}}q$ (see Figure 2).

$$\begin{tikzcd} T^{\mathbf{V}}A \arrow[r] & T^{\mathbf{V}}Q \\ A \arrow[u, "\eta^{\mathbf{V}}_A"] \arrow[r, "q"'] & Q \arrow[u, "\eta^{\mathbf{V}}_Q"'] \end{tikzcd}$$

FIGURE 2. $T^{\mathbf{V}}$ preserves regular epis.

□

Example 3.2.9. **Set** is quasi-Birkhoff. However, the only quasi-varieties of **Set** are trivial. Let $\mathbf{V}$ be a quasi-variety. If $2 \in \mathbf{V}$, then 2^α is in **Set** for every ordinal α. Since $\mathbf{V}$ is closed under subobjects, we have that $\mathbf{V} = \mathbf{Set}$. If $2 \notin \mathbf{V}$, then $\mathbf{V}$ must consist of just 0 and 1.

Example 3.2.10. The category of monoids, **Mon**, is complete, regular and well-powered. Hence, **Mon** is a quasi-Birkhoff category. Let $\mathbf{V}$ be the subcategory of **Mon** consisting of all those monoids satisfying

$$\forall x \in M(x^2 = e \to x = e).$$

Then $\mathbf{V}$ is clearly closed under subalgebras and limits. Thus, by Theorem 3.2.7, $\mathbf{V}$ is a regular epi-reflective subcategory of **Mon**.

3.2.5. The generalized Birkhoff variety theorem. The following may be seen as a generalization of Birkhoff's variety theorem. Recall from Section 3.2.2 that a class $\mathbf{V}$ of Γ-algebras satisfies a set E of equations over a set X of variables just in case $\mathbf{V}$ is orthogonal to a certain regular epi with domain FX. In the following theorem, we show that $\mathbf{V}$ is a Birkhoff variety iff $\mathbf{V}$ is orthogonal to a collection of regular epis with regular projective domains. The regular projective objects play the role of FX (which is regular projective if X is regular projective) in this theorem.

Once we have proven this theorem and shown that it applies to categories of algebras $\mathcal{E}^\Gamma$ (for appropriate base $\mathcal{E}$ and functor Γ), we have still not quite recovered the classical theorem. In particular, we will have shown, essentially, that any variety of algebras $\mathbf{V}$ is definable by a class of equations (i.e., $\mathbf{V} = S^\perp$ for a class of arrows S), rather than by a set of equations. This property is the distinction between Birkhoff categories and *uniformly* Birkhoff categories, which we discuss in Section 3.4.

Theorem 3.2.11. *If $\mathcal{C}$ is a Birkhoff category, then a full subcategory $\mathbf{V}$ is a variety iff $\mathbf{V}^\perp$ is spanned by a collection of regular epis with regular projective domains.*

Proof. Suppose that $\mathbf{V}$ is a variety. Then $\mathbf{V}$ is a regular epi-reflective subcategory of $\mathcal{C}$. Let $F^\mathbf{V} \dashv U^\mathbf{V}$ with unit $\eta^\mathbf{V}$, as in Theorem 3.2.7. For each $C \in \mathcal{C}$, pick a regular epi $p_C : A_C \twoheadrightarrow C$, with A_C regular projective, and let S be the collection of all

$$\eta^\mathbf{V}_{A_C} : A_C \longrightarrow U^\mathbf{V} F^\mathbf{V} A_C .$$

Then $S \subseteq (\eta^\mathbf{V})$ and so $S^\perp \supseteq (\eta^\mathbf{V})^\perp = \mathbf{V}$. To see that $\mathbf{V} = S^\perp$, suppose that $S \perp C$ and we will show that $C \in \mathbf{V}$. Since $S \perp C$, there is a map $\overline{p_C}$ such that

$\overline{p_C} \circ \eta^{\mathbf{V}}_{A_C} = p_C$.

$$\begin{tikzcd}
A_C \arrow[r, twoheadrightarrow, "p_C"] \arrow[d, twoheadrightarrow, "\eta^{\mathbf{V}}_{A_C}"'] & C \\
U^{\mathbf{V}}F^{\mathbf{V}}A_C \arrow[ur, twoheadrightarrow, "\overline{p_C}"']
\end{tikzcd}$$

Since p_C is a regular epi, so is $\overline{p_C}$. Thus, C is a quotient of $U^{\mathbf{V}}F^{\mathbf{V}}A_C$ and hence is in $\mathbf{V}$.

Suppose conversely that $\mathbf{V}^{\perp}$ is spanned by a collection S of regular epis with regular projective domains. Then $\mathbf{V}$ is closed under subobjects and limits (Theorem 3.2.7), so it suffices to show that $\mathbf{V}$ is closed under quotients. Let $K \in \mathbf{V}$ and $p\colon K \twoheadrightarrow K'$ be given. We wish to show that $S \perp K'$. Let $f\colon A \twoheadrightarrow B \in S$ and $g\colon A \to K'$ be given.

$$\begin{tikzcd}
A \arrow[r, "g'"] \arrow[d, twoheadrightarrow, "f"'] \arrow[dr, "g"] & K \arrow[d, twoheadrightarrow, "p"] \\
B \arrow[r, "p\circ\overline{g}"'] & K'
\end{tikzcd}$$

Since A is regular projective, there is a $g'\colon A \to K$ such that $p \circ g' = g$. Since $f \perp K$, there is a unique $\overline{g}\colon B \to K$ such that $\overline{g} \circ f = g'$. Thus,

$$p \circ \overline{g} \circ f = p \circ g' = g.$$

Because f is epi, $p \circ \overline{g}$ is the unique map with this property. ☐

EXAMPLE 3.2.12. Consider the full subcategory **Ab** of **Mon** consisting of abelian monoids. That is, a monoid M is in **Ab** just in case for every m, n in M,

$$m \cdot n = n \cdot m.$$

This subcategory is a variety of **Mon**. It is easy to see that, if M is abelian and N is the homomorphic quotient of M, then N is abelian.

$\mathbf{Ab}^{\perp}$ is spanned by a single regular epimorphism with regular projective domain. Let $F2$ be the free monoid generated by two elements, a and b. Let ab, $ba\colon 1 \to UF2$ be the obvious constant maps. These correspond under adjoint transposition to maps

$$F1 \underset{\overline{ba}}{\overset{\overline{ab}}{\rightrightarrows}} F2.$$

Take the coequalizer $q\colon F2 \twoheadrightarrow Q$ of these homomorphisms. Then a monoid M is evidently in **Ab** iff $q \perp M$.

EXAMPLE 3.2.13. Consider again the full subcategory $\mathbf{V}$ of **Mon** consisting of monoids where no non-unit element is its own inverse (from Example 3.2.10). This

$$\begin{tikzcd}
 & & & T^{\mathbf{V}}/f \\
T^{\mathbf{V}}K \arrow[r, shift left, "T^{\mathbf{V}}k_1"] \arrow[r, shift right, "T^{\mathbf{V}}k_2"'] & T^{\mathbf{V}}A \arrow[r, "T^{\mathbf{V}}q"'] \arrow[urr, bend left, "p"] & T^{\mathbf{V}}Q \arrow[ur, dotted, "\overline{g}"] & \\
K \arrow[u, "\eta^{\mathbf{V}}_K"] \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] & A \arrow[u, "\eta^{\mathbf{V}}_A"] \arrow[r, "q"'] & Q \arrow[u, "\eta^{\mathbf{V}}_Q"] \arrow[uur, dotted, "g"'] &
\end{tikzcd}$$

FIGURE 3. $T^{\mathbf{V}}$ preserves coequalizers.

subcategory is not closed under quotients. For instance, the map

$$p : \mathbb{N} \longrightarrow 2$$

taking even numbers to 0 and odd numbers to 1 is a regular epi in **Mon**, but $2 \notin \mathbf{V}$.

In Corollary 3.2.8, we saw that, if $\mathbf{V}$ is a quasi-variety, then $T^{\mathbf{V}}$ preserves regular epis. We can strengthen that result if $\mathbf{V}$ is a variety.

COROLLARY 3.2.14. *If $\mathbf{V}$ is a variety, then the monad $T^{\mathbf{V}} : \mathcal{C} \to \mathcal{C}$ preserves coequalizers.*

PROOF. Let

$$K \underset{k_2}{\overset{k_1}{\rightrightarrows}} A \xrightarrow{q} Q$$

be a coequalizer. Suppose that $f : T^{\mathbf{V}}A \to B$ coequalizes $T^{\mathbf{V}}k_1$ and $T^{\mathbf{V}}k_2$. Take the regular epi-mono factorization of f, $f = i \circ p$ (see Figure 3). Then

$$\begin{aligned} p \circ \eta^{\mathbf{V}}_A \circ k_1 &= p \circ T^{\mathbf{V}}k_1 \circ \eta^{\mathbf{V}}_K \\ &= p \circ T^{\mathbf{V}}k_2 \circ \eta^{\mathbf{V}}_K \\ &= p \circ \eta^{\mathbf{V}}_A \circ k_2, \end{aligned}$$

so there is a unique map $g : Q \to B$ such that $p \circ \eta^{\mathbf{V}}_A = g \circ q$. Since $T^{\mathbf{V}}A/f \in \mathbf{V}$, the map g factors uniquely through $T^{\mathbf{V}}Q$, say $\overline{g} \circ \eta^{\mathbf{V}}_Q = g$. This factorization gives the desired map

$$T^{\mathbf{V}}Q \xrightarrow{\overline{g}} T^{\mathbf{V}}/f \xrightarrow{i} B.$$

Since $\overline{g} \circ T^{\mathbf{V}}q = p$ and p is a regular epi, so is $\overline{g}$. By the uniqueness of regular epi-mono factorizations, $i \circ \overline{g}$ is the unique map such that $i \circ \overline{g} \circ T^{\mathbf{V}}q = f$. $\square$

3.3. Categories of algebras

In this section, we will show that Theorem 3.2.11 (the abstract variety theorem) is a generalization of the classical variety theorem. To this end, we must first show that categories of algebras $\mathcal{E}^{\Gamma}$ are Birkhoff categories, for suitable base categories $\mathcal{E}$ and endofunctors Γ. It will follow, then, that any Birkhoff variety $\mathbf{V}$ of $\mathcal{E}^{\Gamma}$ satisfies $\mathbf{V} = S^{\perp}$ for some collection S of regular epis with regular projective domains.

The classical theorem says that any variety is an equational variety for some *set* E of equations. If we apply Theorem 3.2.11 to categories $\mathcal{E}^{\Gamma}$, we learn only that each variety is definable by some *class* of equations. To recover the classical theorem, some more work is needed. In Section 3.4.1, we will discuss further conditions on $\mathcal{E}$ and Γ that allow one to conclude that any Birkhoff variety is an equational variety.

The work in this section is similar to work found in [**BH76**] and extended by Andréyka and Németi. A similar approach is also found in [**AR94**].

3.3.1. Categories of algebras are Birkhoff categories. We will first look at some conditions that are sufficient to ensure that a category of algebras is a Birkhoff category, in the sense of Definition 3.2.1. Throughout this section, let $\mathcal{E}$ be an arbitrary category and let Γ be an endofunctor on $\mathcal{E}$. As we will see, it is sufficient that $\mathcal{E}$ is quasi-Birkhoff and Γ preserves regular epis to conclude that $\mathcal{E}^{\Gamma}$ is quasi-Birkhoff.

THEOREM 3.3.1. *If $\mathcal{E}$ is quasi-Birkhoff and Γ preserves regular epis, then $\mathcal{E}^{\Gamma}$ is quasi-Birkhoff. The same claim holds for categories $\mathcal{E}^{\mathbb{T}}$ of algebras over a monad $\mathbb{T}$ that preserves regular epis.*

PROOF. We need to show that $\mathcal{E}^{\Gamma}$ is regularly co-well-powered, complete and has regular epi-mono factorizations.

- $\mathcal{E}^{\Gamma}$ is complete by Theorem 1.2.4 (U creates limits).
- $\mathcal{E}^{\Gamma}$ has regular epi-mono factorizations by Theorem 1.2.13.
- $\mathcal{E}^{\Gamma}$ is regularly co-well-powered since $\mathcal{E}$ is regularly co-well-powered and U preserves regular epis (Corollary 1.2.15).

Since each of the above facts also holds for categories of algebras over a monad, so does this theorem. $\square$

The additional requirement that ensures that $\mathcal{E}^{\Gamma}$ has enough regular projectives (so that $\mathcal{E}^{\Gamma}$ is a Birkhoff category) is natural enough. Given that $\mathcal{E}^{\Gamma}$ is quasi-Birkhoff, we need only the additional assumption that Γ is a varietor (that is, that U is monadic). This assumption is useful for our interpretation of equations and so is reasonable in this setting. However, recent work in [**AP01**] shows how to define equational varieties for categories of algebras without free algebras.

Corollary 3.3.2. *If $\mathcal{E}$ is Birkhoff, Γ preserves regular epis and U has a left adjoint, F, then $\mathcal{E}^{\Gamma}$ is Birkhoff. The same claim holds for categories $\mathcal{E}^{\mathbb{T}}$ of algebras for a regular-epi-preserving monad T.*

Proof. Given $\langle C, \gamma \rangle$ in $\mathcal{E}^{\Gamma}$, let $p : A \twoheadrightarrow C$ be a regular projective covering of C. We will first show that FA is regular projective. Let $f : FA \rightarrow \langle D, \delta \rangle$ and $q : \langle B, \beta \rangle \twoheadrightarrow \langle D, \delta \rangle$ be given (see Figure 4). Because A is regular projective, there

$$
\begin{tikzcd}
UFA \arrow[r, "Uf"] \arrow[dr, dotted, twoheadrightarrow, "Uf^{\#}"] & D \\
A \arrow[u, "\eta_A"] \arrow[r, "\overline{f}"'] & B \arrow[u, twoheadrightarrow, "Uq"']
\end{tikzcd}
$$

Figure 4. The free algebra over a regular projective object is regular projective.

is a map $\overline{f}$ making the square commute. This ensures the existence of $f^{\#}$ making both triangles commute. Hence, FA is regular projective.

All that remains is to show that FA covers $\langle C, \gamma \rangle$. By the adjunction $F \dashv U$, there is a unique map $p^{\#} : FA \rightarrow \langle C, \gamma \rangle$ such that the diagram below commutes.

$$
\begin{tikzcd}
UFA \arrow[dr, twoheadrightarrow, "Up^{\#}"] & \\
A \arrow[u, "\eta_A"] \arrow[r, twoheadrightarrow, "p"'] & C
\end{tikzcd}
$$

Because U reflects regular epis, $p^{\#}$ is a regular epi.

Since only the characteristic property of freeness was used in the above reasoning, and categories of algebras over a monad always have free algebras, the claim holds for $\mathcal{E}^{\mathbb{T}}$ as well. (Alternatively, prove the claim for categories $\mathcal{E}^{\mathbb{T}}$ and use the fact that, given the hypotheses, U is monadic, i.e., $\mathcal{E}^{\Gamma} \equiv \mathcal{E}^{\mathbb{T}}$ for the monad $\mathbb{T}$ induced by $F \dashv U$.) $\square$

Thus, if $\mathcal{E}$ is Birkhoff, Γ preserves regular epis and is a varietor, then Theorem 3.2.11 applies. Hence, a full subcategory $\mathbf{V}$ of $\mathcal{E}^{\Gamma}$ is a Birkhoff variety iff $\mathbf{V}$ is closed, and $\mathbf{V}^{\perp}$ is spanned by a collection of regular epis with regular projective domains. This is not quite sufficient to imply the classical variety theorem, however. For that, we need to show that there is a projective X such that

$$\mathbf{V} \text{ is a Birkhoff variety iff } \mathbf{V} = \{q : FX \twoheadrightarrow \langle Q, \nu \rangle\}^{\perp} \text{ for some regular epi } q.$$

In other words, we need to show that $\mathbf{V}$ is "definable" by a *single* regular epi q with regular projective domain, not a collection of such arrows. For this, we introduce the notion of "uniformly Birkhoff categories" in the next section.

We close this section with a proof that, if $\mathbf{V}$ is a variety over $\mathcal{E}^{\mathbb{T}}$, then $\mathbf{V}$ is also monadic over $\mathcal{E}$. Together with Corollary 3.2.14 ($T^{\mathbf{V}}$ preserves coequalizers), we see that $\mathbf{V} \equiv \mathcal{E}^{\mathbb{T}'}$ for a regular-epi-preserving monad $\mathbb{T}'$ and so is again a Birkhoff category. Hence, the variety theorem again applies, and subvarieties of $\mathbf{V}$ are equationally definable (by, perhaps, a proper class of equations).

THEOREM 3.3.3. *Let $\mathbf{V}$ be a variety of $\mathcal{E}^{\mathbb{T}}$. Then $\mathbf{V}$ is monadic over $\mathcal{E}$, via the evident forgetful functor.*

PROOF. We apply the special adjoint functor theorem (see [**Bor94**, Theorem 4.4.4,Volume 2]). Of course, $U \circ U^{\mathbf{V}}$ has a left adjoint and reflects isomorphisms since both U and $U^{\mathbf{V}}$ do. The functor U creates split coequalizers (since it is monadic) and $U^{\mathbf{V}}$ creates all coequalizers (an easy consequence of Corollary 3.2.14), the composite creates split coequalizers. □

3.4. Uniformly Birkhoff categories

We have shown that, if $\mathcal{E}$ is Birkhoff, Γ preserves regular epis and is a varietor, then $\mathcal{E}^{\Gamma}$ is Birkhoff. Thus, any variety $\mathbf{V}$ is defined by a collection of regular epis with regular projective domains. In terms of equations, this means that any variety $\mathbf{V}$ is defined by a *class* of equations over a *class* of variables. Birkhoff's variety theorem [**Bir35**] says something stronger. Namely, that any variety $\mathbf{V}$ is defined by a set of equations over a countable set of variables.

Categorically, then, we must show that there is a regular projective $X \in \mathcal{E}$ such that, for any variety $\mathbf{V}$, there is a regular epi p with domain X such that $\mathbf{V} = \{p\}^{\perp}$. We state this condition in general terms in the following definition.

DEFINITION 3.4.1. A Birkhoff category $\mathcal{C}$ is *uniformly Birkhoff* if there is a regular projective object $X \in \mathcal{C}$ such that for any variety $\mathbf{V}$, $\mathbf{V} = \{p\}^{\perp}$ for some regular epi p with domain X. The object X is called the *equational domain for $\mathcal{C}$*.

From Theorem 3.2.7, we know that any variety $\mathbf{V}$ satisfies

$$\mathbf{V} = (\eta^{\mathbf{V}})^{\perp}.$$

In a uniformly Birkhoff category, any variety satisfies a stronger condition: namely, that

$$\mathbf{V} = \{\eta^{\mathbf{V}}_X : X \longrightarrow T^{\mathbf{V}} X\}^{\perp},$$

where X is the equational domain for $\mathcal{C}$.

THEOREM 3.4.2. *Let $\mathcal{C}$ be uniformly Birkhoff and let X be the equational domain for $\mathcal{C}$. Let $\mathbf{V}$ be a variety of $\mathcal{C}$, with $\eta^{\mathbf{V}}$ the unit of the evident adjunction $F^{\mathbf{V}} \dashv U^{\mathbf{V}}$. Then $\mathbf{V} = \{\eta^{\mathbf{V}}_X\}^{\perp}$.*

PROOF. Let $\mathbf{V}$ be a variety of $\mathcal{C}$ and let $p : X \twoheadrightarrow Y$ be given such that $\mathbf{V} = \{p\}^{\perp}$. It suffices to show that, for all $A \in \mathcal{C}$, if $\eta_X^{\mathbf{V}} \perp A$, then $p \perp A$. This is clear, since $\eta_X^{\mathbf{V}}$ factors through p. □

The remainder of this section will be devoted to a discussion of conditions that ensure a category of algebras is uniformly Birkhoff. These conditions will be suggested by the original proof of the variety theorem. The conditions are also influenced by the work of [**AR94**], in which the theory of locally finitely presentable categories is developed. It appears, however, that a locally finitely presentable category isn't sufficient for this goal. In one sense, we need a stronger condition: that regular projective objects are colimits of finitely presentable *retracts*. On the other hand, we don't require that every object has a presentation. Instead, it suffices that certain regular projective objects have a retractable presentation in order to show that the category is uniformly Birkhoff.

We recall the following definitions from ibid.

DEFINITION 3.4.3. An object K in $\mathcal{C}$ is *finitely presentable* if the functor

$$\mathsf{Hom}(K, -) : \mathcal{C} \longrightarrow \mathbf{Set}$$

preserves filtered colimits.

DEFINITION 3.4.4. A category $\mathcal{C}$ is *locally finitely presentable* if it is cocomplete and there is a set $\mathcal{A}$ of finitely presentable objects of $\mathcal{C}$ such that every object is a filtered colimit of objects of $\mathcal{A}$.

The remaining work is technical and abstruse. This section is self-contained — there are no later results in this thesis that require the definitions and theorems that follow. The casual reader may wish to skim what remains here.

REMARK 3.4.5. Throughout this section, we use finitely presentable objects and prove facts about filtered colimits in $\mathcal{C}$. This work can be generalized, so that the objects of interest are κ-presentable and the colimits are colimits of κ-filtered diagrams. We avoid the more general statements and proofs in order to present this work in a simpler form.

As we will see, a key step in showing that a category $\mathcal{C}$ is uniformly Birkhoff is showing that every variety of $\mathcal{C}$ is closed under filtered colimits. We first consider the case in which $\mathbf{V} = \{\eta_X^{\mathbf{V}}\}^{\perp}$ where X is finitely presentable. In the classical setting, this corresponds to a variety of algebras which are defined by a set of equations over a *finite* set of variables. Such varieties are easily shown to be closed under filtered colimits. This fact will be used in Theorem 3.4.9, in which we prove that every variety (in a suitable category) is closed under filtered colimits.

LEMMA 3.4.6. *Let $f:X \twoheadrightarrow Q$ be given with X finitely presentable. Then $\{f\}^{\perp}$ is closed under filtered colimits.*

PROOF. Let $A \in \mathcal{C}$ and $K:\mathbb{E} \to \mathcal{C}$ be a filtered diagram such that $A = \mathsf{colim}\ K$ with colimiting cocone

$$k:K \Longrightarrow A.$$

Assume, further, that for each $E \in \mathbb{E}$, $f \perp KE$. We will show that $f \perp A$, so that $A \in \{f\}^{\perp}$.

Let $g:X \to A$ be given. Since X is finitely presentable, there is an $E \in \mathbb{E}$ and a map

$$\overline{g}:X \longrightarrow KE$$

such that $g = k_E \circ \overline{g}$. Hence, there is a unique

$$\widetilde{g}:Q \longrightarrow KE$$

such that $\overline{g} = \widetilde{g} \circ f$ and so $g = k_E \circ \widetilde{g} \circ f$ (see Figure 5). Uniqueness follows from the

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, two heads, "f"'] \arrow[dr, dotted, "\overline{g}"] & A \\
Q \arrow[r, dotted, "\widetilde{g}"'] & KE \arrow[u, "k_E"']
\end{tikzcd}$$

FIGURE 5. $\{f\}^{\perp}$ is closed under filtered colimits.

fact that f is epi. $\square$

We now turn our attention to a special case of a presentation by finitely presentable objects. In this case, we assume that an object is the filtered colimit of finitely presentable retracts, and so this is a stronger condition than that required by a locally presentable category. However, we will not require that every object has such a presentation (see Definition 3.4.8).

The notion of "retractably presentable regular projective" and Theorem 3.4.9 are due to Steve Awodey.

DEFINITION 3.4.7. Let $X \in \mathcal{C}$. We call a filtered diagram $J:\mathbb{D} \to \mathcal{C}$ a *retractable presentation* of C if J satisfies the following:

- $\mathsf{colim}\ J = X$ with cocone $j:J \Rightarrow X$;
- Each JD is finitely presentable;
- Each JD is a retract of X (i.e., for each j_D, there is a $p_D:X \to JD$ such that $p_D \circ j_D = \mathsf{id}_{JD}$).

If there is a retractable presentation of X, then we say that X is *retractably presentable.*

DEFINITION 3.4.8. A category $\mathcal{C}$ *has enough retractably presentable regular projectives* if each object of $\mathcal{C}$ is a quotient of a retractably presentable regular projective object.

It is easy to check that, in a Birkhoff category $\mathcal{C}$ with enough retractably presentable regular projectives, any variety is determined by regular epis with retractably presentable regular projective domains. In fact, if $\mathcal{B}$ is *any* class of regular projectives such that each object of $\mathcal{C}$ is covered by an object in $\mathcal{B}$, then any variety is defined by regular epis with domains in $\mathcal{B}$. Moreover, to confirm that an object A is in a variety $\mathbf{V}$, it suffices to check that $\eta_X^{\mathbf{V}} \perp A$ for some $X \in \mathcal{B}$ covering A.

In a category with enough retractably presentable regular projectives, every variety $\mathbf{V}$ is closed under filtered colimits. This implies that the monad $T^{\mathbf{V}}$ preserves filtered colimits.

THEOREM 3.4.9. *Let $\mathcal{C}$ have enough retractably presentable regular projectives. Let $\mathbf{V}$ be a variety of $\mathcal{C}$. Then $\mathbf{V}$ is closed under filtered colimits.*

PROOF. Let $A \in \mathcal{C}$ and $K:\mathbb{E} \to \mathcal{C}$ be a filtered diagram such that $A = \mathsf{colim}\ K$ with colimiting cocone

$$k: K \Longrightarrow A.$$

Assume, furthermore, that each $KE \in \mathbf{V}$. We will show that $A \in \mathbf{V}$.

Let X be a retractably presentable regular projective which covers A and let $X = \mathsf{colim}\ J$ with cocone j and retractions p, as in Definition 3.4.7. It suffices (by the proof of Theorem 3.2.11) to show that $\eta_X^{\mathbf{V}} \perp A$ to prove $A \in \mathbf{V}$. Let ΘX be the kernel pair of $\eta_X^{\mathbf{V}}$ — so ΘX is the "set" of equations satisfied by $T^{\mathbf{V}} X$.

For each $D \in \mathbb{D}$, take the pullback ΘD as shown below.

$$\begin{tikzcd} \Theta D \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \Theta X \arrow[d, tail] \\ JD \times JD \arrow[r, tail, "j_D \times j_D"'] & X \times X \end{tikzcd}$$

Because $\mathbb{D}$ is filtered, ΘX is the colimit of the ΘD's. Define a functor $Q:\mathbb{D} \to \mathcal{C}$ by taking QD to be the coequalizer of ΘD, as shown in Figure 6. Because colimits commute with coequalizers, $T^{\mathbf{V}} X$ is the colimit of Q.

We next show that

$$\{\eta_X^{\mathbf{V}}\}^{\perp} \subseteq \{q_D \mid D \in \mathbb{D}\}^{\perp}.$$

$$\begin{tikzcd}
\Theta X \arrow[r, shift left] \arrow[r, shift right] & X \arrow[r, "\eta^{\mathbf{V}}_X"] \arrow[d, bend right, "p_D"'] & T^{\mathbf{V}}X \\
\Theta D \arrow[u, "\vartheta_D"] \arrow[r, shift left] \arrow[r, shift right] & JD \arrow[r, "q_D"'] \arrow[u, bend right, "j_D"'] & QD \arrow[u, "\nu_D"']
\end{tikzcd}$$

FIGURE 6. $T^{\mathbf{V}}X$ is the colimit of Q.

Let $\eta^{\mathbf{V}}_X \perp B$ and let $f\colon JD \twoheadrightarrow B$. Let x_1, x_2 (d_1, d_2, resp.) be the projections of ΘX (ΘD, resp.). Then,

$$\begin{aligned}
f \circ d_1 &= f \circ p_D \circ x_1 \circ \vartheta_D \\
&= f \circ p_D \circ x_2 \circ \vartheta_D && \text{(since } \eta^{\mathbf{V}}_X \perp B\text{)} \\
&= f \circ d_2
\end{aligned}$$

and so $q_D \perp B$.

Thus, since each $\eta^{\mathbf{V}}_X \perp KE$ by hypothesis, for each $D \in \mathbb{D}$, we also have $q_D \perp KE$. Now, by definition, each JD is finitely presentable. Thus, by Lemma 3.4.6, A is orthogonal to each q_D. It is routine to check that, since $\mathsf{colim}\ Q = T^{\mathbf{V}}X$ and each $q_D \perp A$, then also $\eta^{\mathbf{V}}_X \perp A$. □

THEOREM 3.4.10. *Let $\mathcal{C}$ be a quasi-Birkhoff category and let $\mathbf{V}$ be a quasi-variety of $\mathcal{C}$ closed under filtered colimits (i.e., the inclusion $\mathbf{V} \hookrightarrow \mathcal{C}$ creates such colimits). Then the monad*

$$T^{\mathbf{V}}\colon \mathcal{C} \longrightarrow \mathcal{C}$$

preserves filtered colimits.

PROOF. Let $\mathbb{E}$ be filtered and let A be the colimit of $K\colon \mathbb{E} \to \mathcal{C}$, with colimiting cocone $k\colon K \Rightarrow A$. Let $j\colon T^{\mathbf{V}}K \Rightarrow B$ be a colimiting cocone. We wish to show that $B \cong T^{\mathbf{V}}A$.

$$\begin{tikzcd}
A \arrow[r, "\eta^{\mathbf{V}}_A"] \arrow[rrd, dotted] & T^{\mathbf{V}}A \arrow[rd, dotted, "m"] & \\
KE \arrow[u, "k_E"] \arrow[r, "\eta^{\mathbf{V}}_{KE}"'] & T^{\mathbf{V}}KE \arrow[u] \arrow[r, "j_E"'] & B \arrow[lu, dotted, bend left, "n"]
\end{tikzcd}$$

Because $B \in \mathbf{V}$, $\eta^{\mathbf{V}}_A \perp B$. Hence there is an $m\colon T^{\mathbf{V}}A \twoheadrightarrow B$ such that $j = m \circ T^{\mathbf{V}}k$. Because j is colimiting, there is an $n\colon B \to T^{\mathbf{V}}A$ such that $n \circ j = T^{\mathbf{V}}k$. It is routine to check that m and n are inverses. □

Let $A = \mathsf{colim}\ J\colon \mathbb{D} \to \mathcal{C}$, where $\mathbb{D}$ is a filtered category and let $\mathbf{V}$ be a variety closed under filtered colimits. Then, in order to check whether an object C is orthogonal to

$\eta^{\mathbf{V}}_A$, it suffices to check that it is orthogonal to each $\eta^{\mathbf{V}}_{JD}$. In the traditional setting, where $\mathcal{C}$ is a category of algebras and $A = FX$ and each $JE = FY$ for some finite Y, this means the following: an algebra $\langle C, \gamma \rangle$ satisfies each of the equations (for $\mathbf{V}$) over X just in case it satisfies each of the equations (for $\mathbf{V}$) over a finite set of variables. We prove this claim in a general setting presently.

Corollary 3.4.11. *Let $\mathcal{C}$ be quasi-Birkhoff and $\mathbb{D}$ be filtered. Let $J\colon \mathbb{D} \to \mathcal{C}$ be given and $X = \mathsf{colim}\ J$ with colimiting cocone $j\colon J \Rightarrow X$. Suppose, further, that $\mathbf{V}$ is closed under filtered colimits. Then*

$$\{\eta^{\mathbf{V}}_{JD} \mid D \in \mathbb{D}\}^{\perp} \subseteq \{\eta^{\mathbf{V}}_X\}^{\perp}.$$

Proof. Let $\{\eta^{\mathbf{V}}_J\} \perp A$ and $f\colon X \to A$ be given. Then, for each $D \in \mathbb{D}$, there is a map $f_D\colon T^{\mathbf{V}}D \to A$ such that $f_D \circ \eta^{\mathbf{V}}_{JD} = f \circ j_D$. The f_D's form a cocone over $T^{\mathbf{V}}J$. Since the colimit of $T^{\mathbf{V}}J$ is $T^{\mathbf{V}}X$, we have the factorization of f through $\eta^{\mathbf{V}}_X$, as desired. $\square$

Thus far, we have discussed a condition that ensures that every Birkhoff variety is closed under filtered colimits. While this is a step towards proving that a category is uniformly Birkhoff, there is still some work to be done. Specifically, given a Birkhoff category $\mathcal{C}$ with enough retractably presentable regular projectives, we must pick out a particular object X that will serve as an equational domain. The theorem below shows sufficient conditions for X to be an equational domain. These conditions are attained in **Set**, for instance, with $X = \mathbb{N}$.

The following lemma is the dual of Lemma 3.7.21. We prove it in Section 3.7.2.

Lemma 3.4.12. *Let $\mathbf{V}$ be a variety in the Birkhoff category $\mathcal{C}$ and let A be a quotient of B. Then*

$$\{\eta^{\mathbf{V}}_B\}^{\perp} \subseteq \{\eta^{\mathbf{V}}_A\}^{\perp}.$$

Theorem 3.4.13. *Let $\mathcal{C}$ be Birkhoff and have enough retractably presentable regular projectives and let X satisfy the following:*

- *X is regular projective;*
- *The set of non-empty, finitely presentable objects is a retractable presentation for X.*

Then $\mathcal{C}$ is uniformly Birkhoff and X is the equational domain for $\mathcal{C}$.

Proof. Let $\mathcal{B}$ be the set of non-empty, finitely presentable objects, so that $\mathcal{B}$ is a retractable presentation of X.

Let $\mathbf{V}$ be a variety of $\mathcal{C}$. Let $A \in \mathcal{C}$ and $\eta^{\mathbf{V}}_X \perp A$. We will show that $A \in \mathbf{V}$. It suffices to show that $\eta^{\mathbf{V}}_Y \perp A$ for a retractably presentable regular projective Y covering A. Let $J\colon \mathbb{D} \to \mathcal{C}$ be a retractable presentation for Y. Then, for each

$D \in \mathbb{D}$, JD is a quotient of X and so $\eta^{\mathbf{V}}_{JD} \perp A$ (Lemma 3.4.12). Thus, $\eta^{\mathbf{V}}_{Y} \perp A$ (Corollary 3.4.11). □

3.4.1. Uniformly Birkhoff categories of algebras. The preceding section demonstrated sufficient conditions for an abstract category to be uniformly Birkhoff. In this section, we show that, if $\mathcal{E}$ satisfies these conditions and Γ preserves regular epis and filtered colimits (more generally, κ-filtered colimits), then $\mathcal{E}^{\Gamma}$ is also uniformly Birkhoff. This will conclude the reconstruction of the classical Birkhoff variety theorem in a categorical setting.

In particular, the category **Set** satisfies the conditions of Theorem 3.4.13 (so **Set** is uniformly Birkhoff). Thus, if $\Gamma : \mathbf{Set} \to \mathbf{Set}$ preserves filtered colimits, then $\mathbf{Set}^{\Gamma}$ is uniformly Birkhoff. Moreover, the free algebra over a countable set is an equational domain for $\mathbf{Set}^{\Gamma}$. In other words, if Γ preserves filtered colimits, then any variety of $\mathbf{Set}^{\Gamma}$ is definable by a set of equations over a countable set of variables (which is, of course, just the classical Birkhoff theorem as found in [**Bir35**]).

We begin by showing that if X is finitely presentable and Γ preserves filtered colimits, then the free algebra FX is finitely presentable. Hence, applying Lemma 3.4.6, we see that any $\mathbf{V}$ defined by a set of equations E over finitely presentable X is closed under filtered colimits.

LEMMA 3.4.14. *Let $\Gamma : \mathcal{E} \to \mathcal{E}$ preserve filtered colimits and be a varietor with F left adjoint to $U : \mathcal{E}^{\Gamma} \to \mathcal{E}$. If $X \in \mathcal{E}$ is finitely presentable then FX is finitely presentable.*

PROOF. Let $\langle A, \alpha \rangle = \mathsf{colim}\ K : \mathbb{E} \to \mathcal{E}^{\Gamma}$, $\mathbb{E}$ filtered, and $f : FX \to \langle A, \alpha \rangle$. Then

$$UA = \mathsf{colim}\ UK$$

and so the adjoint transpose $\widetilde{f} : X \to A$ of f factors through some UKE. Thus, f factors through KE. □

Lemma 3.4.14 ensures that $\mathcal{E}^{\Gamma}$ inherits the relevant structure (for Theorem 3.4.13) from $\mathcal{E}$. In particular, one shows that if X is a retractably presentable regular projective, then so is FX (under the assumptions of Lemma 3.4.14). From this, it follows that $\mathcal{E}^{\Gamma}$ has enough retractably presentable regular projectives whenever $\mathcal{E}$ does. We show this in Theorem 3.4.15, which directly implies Birkhoff's variety theorem for universal algebras. This completes the categorical approach to the 1935 theorem.

THEOREM 3.4.15. *Let $\mathcal{E}$, X satisfy the conditions of Theorem 3.4.13. Let Γ be preserve regular epis and filtered colimits. Then $\mathcal{E}^{\Gamma}$ is uniformly Birkhoff and FX is an equational domain for $\mathcal{E}^{\Gamma}$.*

PROOF. Let $\mathcal{B}$ be the category of non-empty, finitely presentable objects of $\mathcal{E}$, so $X = \mathsf{colim}\ \mathcal{B}$. Then FX is the colimit of $F\mathcal{B}$ (left adjoints preserve limits), and for each $B \in \mathcal{B}$, FB is finitely presentable. Furthermore, each FB is a quotient of FX, since F preserves coequalizers.

For each $\langle A, \alpha\rangle \in \mathcal{E}^{\Gamma}$, there is a retractably presentable regular projective Y_A such that FY_A covers $\langle A, \alpha\rangle$. Furthermore, each FY_A has a retractable presentation using the objects of $F\mathcal{B}$. Now, $F\mathcal{B}$ is not the collection of *all* finitely presentable algebras. Nonetheless, a simple alteration of the proof of Theorem 3.4.13 using the above facts shows that FX is an equational domain (for our purposes, $F\mathcal{B}$ is a *sufficient* collection of finitely presentable algebras). □

EXAMPLE 3.4.16. If $\Gamma : \mathbf{Set} \to \mathbf{Set}$ preserves filtered colimits (for instance, if Γ is a polynomial), then $F\mathbb{N}$ is a presentational domain for $\mathbf{Set}^{\Gamma}$. Consequently, $F\mathbb{N}$ is an equational domain for $\mathbf{Set}^{\Gamma}$. In other words, every variety in $\mathbf{Set}^{\Gamma}$ is defined by a set of equations over a countable set of variables.

We finish this section by showing that Theorem 3.4.17 does indeed yield the traditional statement of Birkhoff's theorem. To do this, we first recall the definitions from Section 3.2.2

An *equation* over X is a pair of global elements

$$1 \rightrightarrows UFX$$

of UFX and that a *set of equations* over X is given by a pair of maps

$$E \underset{e_2}{\overset{e_1}{\rightrightarrows}} UFX \ .$$

An algebra $\langle A, \alpha\rangle$ satisfies E just in case, for every homomorphism

$$\widetilde{\sigma} : FX \longrightarrow \langle A, \alpha\rangle \, ,$$

$U\widetilde{\sigma}$ coequalizes e_1 and e_2. Equivalently, $\langle A, \alpha\rangle \models E$ just in case $q \perp \langle A, \alpha\rangle$, where q is the coequalizer of

$$FE \rightrightarrows FX \ .$$

With these definitions in mind, it is easy to see that Birkhoff's variety theorem is a corollary to Theorem 3.4.15.

THEOREM 3.4.17. *Let $\mathcal{E}$ be a Birkhoff category and Γ preserve regular epis and is a varietor. Suppose, further, that the category $\mathcal{E}^{\Gamma}$ is uniformly Birkhoff, with equational domain FX. A full subcategory $\mathbf{V}$ of $\mathcal{E}^{\Gamma}$ is a variety iff there is a set E of equations over X such that*

$$\langle A, \alpha\rangle \in \mathbf{V} \textit{ iff } \langle A, \alpha\rangle \models E.$$

PROOF. Let $\mathbf{V}$ be a variety and $\eta^{\mathbf{V}}$ the unit of the adjunction $F^{\mathbf{V}} \dashv U^{\mathbf{V}}$. Let ΘX be the kernel pair of $\eta^{\mathbf{V}}$. Then

$$\langle A, \alpha\rangle \in \mathbf{V} \text{ iff } \langle A, \alpha\rangle \models U\Theta X.$$

□

3.5. Deductive closure

We continue developing the results of Section 3.4.1. Consequently, throughout we assume that $\mathcal{E}$ is a Birkhoff category, Γ preserves regular epis and is a varietor. Also, we fix a regular projective $X \in \mathcal{E}$. For another presentation of this material and the material of Section 3.8, see [**Hug01**].

Birkhoff's variety theorem may be viewed as showing an equivalence between equational definability on the one hand and closure under the operators H, S and P from Section 3.2.1 on the other. When we say that a class $\mathbf{V}$ is equationally definable (over the fixed set X of "variables"), we mean that there is a set E of equations over X such that $\mathbf{V}$ consists of just those algebras which satisfy E. This suggests an operator

$$\mathcal{S}at_X : \mathsf{Rel}(UFX, UFX) \longrightarrow \mathsf{Sub}(\mathcal{E}^\Gamma),$$

taking a set E of equations to the variety

$$\mathcal{S}at_X(E) = \{\langle A, \alpha\rangle \in \mathcal{E}^\Gamma \mid \langle A, \alpha\rangle \models E\}$$

(hereafter, we omit the subscript). In other words, if q is the coequalizer of the diagram

$$FE \rightrightarrows FX,$$

then $\mathcal{S}at(E) = \{q\}^\perp$. In these terms, Theorem 3.4.17 says that, for any class $\mathbf{V}$ of algebras,

$$\mathbf{V} = HSP\mathbf{V}$$

just in case there is some $E \leq UFX \times UFX$ such that

$$\mathbf{V} = \mathcal{S}at(E).$$

One may ask whether there is an analogous result for sets of equations. That is, given a set E of equations, when does E consist of exactly those equations which hold in some variety $\mathbf{V}$?

More precisely, we define an operator

$$\mathcal{I}d_X : \mathsf{Sub}(\mathcal{E}^\Gamma) \longrightarrow \mathsf{Rel}(UFX, UFX)$$

(hereafter, omitting the subscript) taking a class of algebras $\mathbf{V}$ to the set of equations

$$\{e_1 = e_2 \mid \mathbf{V} \models e_1 = e_2\}.$$

In terms of the $\perp$ operators from Section 3.2.3,

$$\mathcal{I}d\,\mathbf{V} = \bigcup\{\ker(f : FX \longrightarrow \bullet) \mid f \in \mathbf{V}^\perp\}.$$

Notice that the operators $\mathcal{I}d$ and $\mathcal{S}at$ form a Galois correspondence. That is, for all classes of algebras $\mathbf{V}$ and sets of equations E, we have

$$\mathbf{V} \leq \mathcal{S}at(E) \text{ iff } \mathcal{I}d(V) \geq E.$$

REMARK 3.5.1. The operators $\mathcal{S}at$ and $\mathcal{I}d$ could be defined for any algebra $\langle A, \alpha \rangle$, of course, and not just the free algebras FX. We focus on the free algebras here for their importance in the completeness and variety theorems.

We would like to find conditions on E that ensure $E = \mathcal{I}d(\mathbf{V})$ for some class $\mathbf{V}$ of algebras. Birkhoff's completeness theorem [**Bir35**] provides that condition.

Classically, given a signature Σ, a set E of equations over X is *deductively closedclosed!deductively* – if it satisfies the following:

(i) For each $x \in X$, $x = x \in E$;
(ii) If $\tau_1 = \tau_2 \in E$, then $\tau_2 = \tau_1 \in E$;
(iii) If $\tau_1 = \tau_2 \in E$ and $\tau_2 = \tau_3 \in E$, then $\tau_1 = \tau_3 \in E$;
(iv) If $f^{(n)} \in \Sigma$ and $\tau_1 = \upsilon_1, \tau_2 = \upsilon_2, \ldots, \tau_n = \upsilon_n$ are in E, then $f^{(n)}(\tau_1, \tau_2, \ldots, \tau_n) = f^{(n)}(\upsilon_1, \upsilon_2, \ldots, \upsilon_n) \in E$.
(v) For any assignment of variables $\sigma : X \to UFX$, if $\tau_1 = \tau_2 \in E$, then $\widetilde{\sigma}(\tau_1) = \widetilde{\sigma}(\tau_2) \in E$.

THEOREM (Birkhoff's completeness theorem). *Let E be a set of equations. Then $E = \mathcal{I}d(\mathbf{V})$ for some class of algebras $\mathbf{V}$ iff E is deductively closed.*

We can restate the definition of deductive closure in categorical terms (and, in particular, eliminate the reference to function symbols in (iv)). For this, we require a definition.

DEFINITION 3.5.2. Let $E \rightrightarrows UFX$ be a set of equations over X. We say that E is *endomorphism-stable* (or just *stable*) if, for every homomorphism

$$\sigma : FX \longrightarrow FX,$$

there is a (necessarily unique) map $\psi : E \to E$ such that the diagram below commutes.

$$\begin{tikzcd} E \arrow[r, dotted, "\psi"] \arrow[d, shift left] \arrow[d, shift right] & E \arrow[d, shift left] \arrow[d, shift right] \\ UFX \arrow[r, "U\sigma"'] & UFX \end{tikzcd}$$

More generally, if

$$E \overset{e_1}{\underset{e_2}{\rightrightarrows}} A$$

is a relation over the carrier of a algebra $\langle A, \alpha \rangle$, we say that E is stable if, for every Γ-endomorphism $\phi : \langle A, \alpha \rangle \to \langle A, \alpha \rangle$, there is a map $\psi : E \to E$ such that $\phi \circ e_1 = e_1 \circ \psi$ and $\phi \circ e_2 = e_2 \circ \psi$.

Let E be a relation over UFX. Then E is *closed!deductively –deductively closed* just in case the following hold.

(i′) $\langle \eta_X, \eta_X \rangle$ factors through E;
(ii′) E is symmetric;
(iii′) E is transitive;
(iv′) E is (the carrier of) a pre-congruence;
(v′) E is stable.

We first show a couple of easy theorems about the relationship between deductive completeness and stable congruences. The first theorem relates orthogonality to stability, and so ties up some of the previous work with this section. The theorem thereafter shows that stable congruences over FX just are the sets of deductively closed equations — an easy consequence. Following this, we show that, given an equationally defined variety $\mathbf{V}$, the set of equations $\mathcal{I}d(\mathbf{V})$ is exactly the kernel of $\eta^{\mathbf{V}}_{FX}$.

Theorem 3.5.3. *Let*

$$E \overset{e_1}{\underset{e_2}{\rightrightarrows}} UFX$$

be a set of equations over regular projective X, and $q : FX \twoheadrightarrow \langle Q, \nu \rangle$ the coequalizer of

$$FE \rightrightarrows FX.$$

If E is stable then $q \perp \langle Q, \nu \rangle$. Conversely, if E is the kernel pair of q and $q \perp \langle Q, \nu \rangle$, then E is stable.

Proof. Suppose that E is stable and let $f : FX \twoheadrightarrow \langle Q, \nu \rangle$ be given. Because FX is regular projective, there is a map $\sigma : FX \twoheadrightarrow FX$ such that $f = q \circ \sigma$. Let $\phi : E \twoheadrightarrow E$ be given as in Definition 3.5.2. Then, a simple diagram chase through Figure 7 confirms that $q \circ \sigma$ coequalizes $\widetilde{e}_1$ and $\widetilde{e}_2$, yielding the desired map ψ, as shown.

$$\begin{tikzcd}
FE \arrow[r, shift left, "\widetilde{e}_1"] \arrow[r, shift right, "\widetilde{e}_2"'] \arrow[d, dotted, "F\phi"'] & FX \arrow[r, "q"] \arrow[d, "\sigma"'] \arrow[dr, "f"] & \langle Q, \nu \rangle \arrow[d, dotted, "\psi"] \\
FE \arrow[r, shift left, "\widetilde{e}_1"] \arrow[r, shift right, "\widetilde{e}_2"'] & FX \arrow[r, "q"'] & \langle Q, \nu \rangle
\end{tikzcd}$$

Figure 7. E is stable iff $q \perp \langle Q, \nu \rangle$.

Conversely, suppose that E is the kernel pair of q and $q \perp \langle Q, \nu \rangle$. Let

$$\sigma : FX \longrightarrow FX$$

be given. Since $q \perp \langle Q, \nu \rangle$, there is a unique

$$\psi : \langle Q, \nu \rangle \longrightarrow \langle Q, \nu \rangle$$

such that $q \circ \sigma = \psi \circ q$. Hence, q coequalizes $\sigma \circ e_1$ and $\sigma \circ e_2$ and so, there is a unique map

$$\phi : E \longrightarrow E$$

as desired. □

THEOREM 3.5.4. *A set of equations E is deductively closed iff E is a stable congruence.*

PROOF. E is a stable congruence if and only if, in addition to Conditions (ii') - (v'), E is reflexive (i.e., the diagonal arrow $\Delta_{UFX} : UFX \to UFX \times UFX$ factors through $E \rightarrowtail UFX \times UFX$. If E is reflexive, then clearly $\langle \eta_X, \eta_X \rangle$ factors through E and so Condition (i') is satisfied.

$$\begin{tikzcd}
UFX \arrow[r, "\delta_{UFX}"] \arrow[dr, dotted] & UFX \times UFX \\
X \arrow[u, "\eta_X"] \arrow[r] & E \arrow[u, tail]
\end{tikzcd}$$

FIGURE 8. If E is deductively closed, then it is a stable congruence.

On the other hand, suppose that E is deductively closed. By (i'), $\langle \eta_X, \eta_X \rangle$ factors through E, as shown in Figure 8. By (iv'), there is a structure map $\varepsilon : E \to \Gamma E$ such that $\langle E, \varepsilon \rangle$ is a relation over FX in $\mathcal{E}^\Gamma$. Consequently, there is a unique homomorphism $FX \to \langle E, \varepsilon \rangle$ making the lower triangle commute, as shown. It is easy to confirm that the upper triangle also commutes and thus that E is reflexive.. □

As one would expect, if $\mathbf{V}$ is defined by a set of equations over X, then $\eta^{\mathbf{V}}_{FX}$ is just the coequalizer of $\mathcal{I}d(\mathbf{V})$. This shows the connection between the work in previous sections and the current approach in terms of deductive completeness.

LEMMA 3.5.5. *For any variety $\mathbf{V}$ of the form $\mathbf{V} = \mathcal{S}at(E)$ for some set of equations E over X, $\mathcal{I}d(\mathbf{V}) = \mathsf{ker}\, \eta^{\mathbf{V}}_{FX}$.*

PROOF. Since $\eta^{\mathbf{V}}_{FX} \in \mathbf{V}^\perp$ and $\mathcal{I}d\,\mathbf{V} = \bigcup \{\ker(f : FX \to \bullet) \mid f \in \mathbf{V}^\perp\}$, we see $\mathsf{ker}\, \eta^{\mathbf{V}}_{FX} \leq \mathcal{I}d(\mathbf{V})$. Conversely, since $T^{\mathbf{V}} FX \in \mathbf{V}$, it is orthogonal to each $f \in \mathbf{V}^\perp$. Consequently, each $\ker f$ factors through $\ker \eta^{\mathbf{V}}_{FX}$ and, hence, so does $\mathcal{I}d(\mathbf{V})$. □

Theorem 3.5.6 is the categorical version of Birkhoff's deductive completeness theorem.

THEOREM 3.5.6. *Let Γ preserve exact sequences, so $\mathcal{E}^\Gamma$ is exact. Let*

$$E \overset{e_1}{\underset{e_2}{\rightrightarrows}} UFX$$

be a set of equations over X. Then $E = \mathcal{I}d(\mathbf{V})$ for some class $\mathbf{V}$ of algebras iff E is a stable congruence.

PROOF. Let $E = \mathcal{I}d(\mathbf{V})$ for some class $\mathbf{V}$ of algebras. By the Galois correspondence $\mathcal{I}d \dashv \mathcal{S}at$, we know that $E \leq \mathcal{I}d\,\mathcal{S}at(E)$. Since $\mathbf{V} \leq \mathcal{S}at(E)$, we also have $\mathcal{I}d\,\mathcal{S}at(E) \leq \mathcal{I}d(\mathbf{V}) = E$. Thus, $E = \mathcal{I}d\,\mathcal{S}at(E)$. Since $\mathcal{S}at(E)$ is a variety, we can make use of the work of the preceding sections. Let $\mathbb{T}^E : \mathcal{E}^\Gamma \to \mathcal{E}$ be the associated monad, with unit η^E.

By Lemma 3.5.5, $E = \mathsf{ker}\,\eta^E_{FX}$. Hence, in particular, E is a congruence. Let $\sigma : X \to UFX$ be given. Since $\mathbb{T}^E FX \in \mathcal{S}at(E)$, η^E_{FX} coequalizes the composite $\widetilde{\sigma} \circ e_1$ and $\widetilde{\sigma} \circ e_2$. Because E is the kernel pair of η^E_{FX}, we have the factorization $E \to E$, as desired. Hence, E is stable.

Let E be a stable congruence and let $q : FX \to \langle Q, \nu\rangle$ be the coequalizer of

$$FE \rightrightarrows FX\ .$$

Let $\mathbf{V} = q^\perp$ (i.e., $\mathbf{V} = \mathcal{S}at(E)$). Because E is stable, $q \perp \langle Q, \nu\rangle$, so $\langle Q, \nu\rangle \in \mathbf{V}$. Hence, $\langle Q, \nu\rangle \cong T^{\mathbf{V}}FX$. Since E is a congruence, it is the kernel pair of its coequalizer. Thus, by Lemma 3.5.5, $\mathcal{I}d(\mathbf{V}) = E$. $\square$

REMARK 3.5.7. Theorem 3.5.6 applies more generally than stated. If E is a relation over $A = U\langle A, \alpha\rangle$ (not necessarily free), then $E = \mathcal{I}d(\mathbf{V})$ for some class $\mathbf{V}$ of algebras iff E is a stable congruence. In this more general case, $\mathcal{S}at(E)$ is, of course, a quasi-variety rather than a variety.

COROLLARY 3.5.8. *For any set E of equations over regular projective X,*

$$\mathcal{I}d(\mathcal{S}at(E)) = \ker \eta^E_{FX},$$

where η^E is defined as in the proof of Theorem 3.5.6.

THEOREM 3.5.9. *Assume that $\mathcal{E}^\Gamma$ is uniformly Birkhoff, with FX an equational domain. Let* Var *be the collection of varieties of $\mathcal{E}^\Gamma$, ordered by reverse inclusion and let* Ded *be the collection of stable congruences over FX (an equational domain), ordered by inclusion. Then* $\mathsf{Ded} \cong \mathsf{Var}$.

PROOF. Var is the collection of fixed points of $\mathcal{S}at \circ \mathcal{I}d$, while Ded is the collection of fixed points of $\mathcal{I}d \circ \mathcal{S}at$. The functors $\mathcal{I}d$ and $\mathcal{S}at$ are isomorphisms when restricted to $\mathsf{Fix}(\mathcal{S}at \circ \mathcal{I}d)$ and $\mathsf{Fix}(\mathcal{I}d \circ \mathcal{S}at)$, respectively. $\square$

The isomorphism between varieties and stable congruences is a result of the isomorphism between congruences and coequalizers in $\mathcal{E}^{\Gamma}$. We stated Theorem 3.5.9 in terms of stable congruences (i.e., deductively closed sets of equations) in keeping with the historical motivation and with the traditional notion of "equationally defined class". However, one has the same result, more or less, in an abstract uniformly Birkhoff category. We sketch the theorem here.

Let $\mathcal{C}$ be a uniformly Birkhoff category with equational domain X (i.e., for any variety $\mathbf{V}$, $\mathbf{V} = \{\eta^{\mathbf{V}}\}^{\perp}$). Call a quotient $q : X \twoheadrightarrow Q$ *stable* if $q \perp Q$. In other words, q is stable just in case $Q \in \{q\}^{\perp}$. If $\mathbf{V} = \{q\}^{\perp}$, then, $q : X \twoheadrightarrow Q$ is stable just in case $Q \cong \mathbb{T}^{\mathbf{V}} X$ (and $q \cong \eta^{\mathbf{V}}_X$).

The quotients of X may be partially ordered by $Q \leq Q'$ if there is a (necessarily regular epi) $Q \twoheadrightarrow Q'$ in $X/\mathcal{C}$. The resulting order StQ of stable quotients of X is isomorphic to the collection Var of varieties in $\mathcal{C}$, ordered by reverse inclusion. The isomorphism takes a stable q to $\{q\}^{\perp}$, while the inverse takes a variety $\mathbf{V}$ to the unit $\eta^{\mathbf{V}}_X$.

.

3.6. The coalgebraic dual of Birkhoff's variety theorem

We now consider the dual of Birkhoff's theorem in categories of coalgebras. To begin, we dualize the definitions of Birkhoff, variety, etc., to prove the dual of Theorem 3.2.11. Following this, we show how this theorem applies to categories of coalgebras.

The co-Birkhoff theorem has been a hot topic lately, beginning with Jan Rutten's co-Birkhoff theorem for **Set** ([**Rut96**]). Peter Gumm and Tobias Schröder continued developing the co-Birkhoff theorem over **Set** in [**GS98**]. The following material essentially dualizes the work done in the previous sections, so coequation satisfaction is again an orthogonality condition (formally dual to equation satisfaction). It can be seen as an extension of the work in [**BH76**], further developed in the papers of Andréyka and Németi [**AN83, Ném82, AN81a, AN81b, AN79a, AN79b, AN78**], discovered by the author after this work was completed independently. The same approach was taken by Alexander Kurz in his dissertation [**Kur00**], again independently of the author.

3.6.1. The dual definitions. Here, we give the dual of the relevant definitions in Section 3.2.1. This is straightforward, but we will explicitly state the definitions here.

An arbitrary category is a *quasi-co-Birkhoff category* if it is regularly well-powered, cocomplete and has epi-regular mono factorizations. Recall that a category *has* *enough regular injectives* if every object is a regular subobject of an regular injective object. If, in addition, the category has enough regular injectives, then it is a *category!co-Birkhoff*.

EXAMPLE 3.6.1. Any co-complete topos $\mathcal{E}$ is co-Birkhoff. That it is regularly well-powered and has epi-regular mono factorizations is clear. Because each object $C \in \mathcal{E}$ has a mono $C \rightarrowtail \mathcal{P}C$ and $\mathcal{P}C$ is regular injective, $\mathcal{E}$ has enough regular injectives [**LM92**, Corollary IV.10.3].

EXAMPLE 3.6.2. **Top** is a co-Birkhoff category. It is obvious that **Top** is regularly well-powered, since monos in **Top** are regular injective functions. Also, every space $\langle X, \mathcal{O}\rangle$ is a regular subobject of an regular injective space. Namely, we can take the space X and adjoin a single point whose singleton is open. This new space is regular injective. That **Top** is cocomplete and has epi-regular mono factorizations is well-known (a regular mono in **Top** is an embedding).

In Theorem 3.6.7, we will show that, given the category $\mathcal{E}$ is co-Birkhoff and that Γ preserves regular monos, then the category $\mathcal{E}_\Gamma$ is also co-Birkhoff.

The dual of Birkhoff's (quasi-)variety theorem will state an equivalence between subcategories satisfying certain closure conditions and class of objects that are orthogonal to some collection of arrows. The closure conditions are easily found: they are the dual of the defining properties of (quasi-)varieties. Consequently, we say that a full subcategory is a *quasi-covariety* if it is closed under codomains of epimorphisms and coproducts and it is a *covariety* if it is also closed under regular subobjects.

An object X *is orthogonal to a map* $f\colon A \rightarrowtail B$ (written $X \perp f$ — sometimes this condition is stated as, f *is co-orthogonal to* X) if, for each $b\colon X \rightarrow B$, b factors through f. In particular, if f is an equalizer for e_1, $e_2\colon B \rightarrow C$, then $X \perp f$ iff every map $X \rightarrow B$ equalizes e_1 and e_2. If S is a class of arrows and $\mathbf{V}$ is a full subcategory, we define the notations $X \perp S$, $\mathbf{V} \perp f$ and $\mathbf{V} \perp S$, as before. The class of arrows $\mathbf{V}_\perp$ consists of all maps f such that $\mathbf{V} \perp f$ and the full subcategory $S_\perp$ consists of all those objects X such that $X \perp S$. These operators form a Galois correspondence.

3.6.2. The abstract dual to Birkhoff's theorem. We can now dualize the theorems of Section 3.2.5, providing quasi-covariety and covariety theorems for abstract co-Birkhoff categories. These theorems will then be interpreted in categories of coalgebras $\mathcal{E}_\Gamma$ for co—Birkhoff $\mathcal{E}$ and covarietor Γ that preserves regular monos, leading to a definition of coequation for such categories.

Recall that $\mathbb{G} = \langle G, \varepsilon, \delta\rangle$ is a comonad just in case G is an endofunctor and ε (the *counit*) and δ (the *comultiplication*) are natural transformations

$$\varepsilon : G \longrightarrow 1_{\mathcal{C}},$$
$$\delta : G \longrightarrow GG,$$

respectively, satisfying $\varepsilon_G \circ \delta = \mathsf{id}_G = G\varepsilon \circ \varepsilon$ and $\delta_G \circ \delta = G\delta \circ \delta$.

The following theorem is the dual of Theorem 3.2.7.

THEOREM 3.6.3. *Let $\mathcal{C}$ be a quasi-co-Birkhoff category and $\mathbf{V}$ a full subcategory of $\mathcal{C}$. The following are equivalent.*

(1) $\mathbf{V}$ is a quasi-covariety.
(2) The inclusion $U^{\mathbf{V}} : \mathbf{V} \to \mathcal{C}$ has a right adjoint $H^{\mathbf{V}}$ such that each component of the counit $\varepsilon^{\mathbf{V}} : 1_{\mathcal{C}} \to U^{\mathbf{V}} H^{\mathbf{V}}$ is a regular mono.
(3) $\mathbf{V} = S_{\perp}$ for some collection S of regular monos.

EXAMPLE 3.6.4. **Top** has no interesting quasi-covarieties. Let $\mathbf{V}$ be a quasi-covariety of topological spaces and suppose that there is a non-empty space A in $\mathbf{V}$. Then, since the space 1 is the codomain of an epi out of A, $1 \in \mathbf{V}$. Hence, so is every discrete space (as a coproduct of 1) and hence every topological space (since injects into the discrete space with the same underlying set).

Unfortunately, we don't have any good examples of covarieties or quasi-covarieties outside of categories of coalgebras yet.

COROLLARY 3.6.5. *Let $\mathcal{C}$ be a quasi-co-Birkhoff category and $\mathbf{V}$ a quasi-covariety of $\mathcal{C}$. Then*

(1) The inclusion $U^{\mathbf{V}} : \mathbf{V} \to \mathcal{C}$ has a right adjoint $H^{\mathbf{V}}$.
(2) The unit $\eta^{\mathbf{V}} : 1_{\mathbf{V}} \to H^{\mathbf{V}} U^{\mathbf{V}}$ is an isomorphism.
(3) For each $C \in \mathcal{C}$, we have $C \in \mathbf{V}$ iff $C \perp \varepsilon^{\mathbf{V}}_C$, where $\varepsilon^{\mathbf{V}}$ is the counit of the adjunction $U^{\mathbf{V}} \dashv H^{\mathbf{V}}$.
(4) The corresponding comonad, $\mathbb{G}^{\mathbf{V}} = \langle U^{\mathbf{V}} H^{\mathbf{V}}, \varepsilon^{\mathbf{V}}, \delta^{\mathbf{V}}\rangle$, is idempotent.
(5) The comonad $\mathbb{G}^{\mathbf{V}}$ preserves regular monos.

Thus, if $\mathbf{V}$ is a quasi-variety of $\mathcal{C}$, we have an adjunction,

$$\begin{tikzcd} \mathbf{V} \arrow[r, bend left, "U^{\mathbf{V}}"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "H^{\mathbf{V}}"] \end{tikzcd},$$

where $U^{\mathbf{V}}$ is full and faithful and every component of the counit

$$\varepsilon^{\mathbf{V}} : U^{\mathbf{V}} H^{\mathbf{V}} \longrightarrow 1$$

is a regular mono, while every component of the unit

$$\eta^{\mathbf{V}} : 1 \longrightarrow H^{\mathbf{V}} U^{\mathbf{V}}$$

is an isomorphism ([**Bor94**, Proposition 3.4.1, Volume 1]). Also, we have that any object $C \in \mathcal{C}$ is in $\mathbf{V}$ iff $C \perp \varepsilon_C^{\mathbf{V}}$, in which case $\varepsilon_C^{\mathbf{V}}$ is an isomorphism. From this, it follows that the comonad $\mathbb{G}^{\mathbf{V}}$ is idempotent. In addition, the comonad $\mathbb{G}^{\mathbf{V}}$ preserves regular monos, by the dual of Corollary 3.2.8, Item (5).

Theorem 3.6.3 provides a quasi-variety theorem which we will interpret in terms of conditional coequations in Section 3.6.4. The following theorem is the formal dual of the abstract Birkhoff theorem, Theorem 3.2.11. This is the theorem which, when interpreted in categories $\mathcal{E}_\Gamma$, yields the co-Birkhoff theorem for categories of coalgebras.

THEOREM 3.6.6. *If $\mathcal{C}$ is a co-Birkhoff category, then $\mathbf{V}$ is a covariety iff $\mathbf{V} = S_\perp$ for some collection S of regular monos with regular injective codomains.*

3.6.3. Covarieties of coalgebras. The formal dualities of Sections 3.6.1 and 3.6.2 provide the basic background for the co-Birkhoff theorem, but our work is not complete. In this section, we will show that categories of coalgebras over co-Birkhoff categories are again co-Birkhoff, and also provide a definition of coequation and coequation satisfaction to provide an interpretation of the co-Birkhoff theorem in $\mathcal{E}_\Gamma$.

That categories $\mathcal{E}_\Gamma$ are co-Birkhoff follows, as before, by duality (of Theorem 3.3.1).

THEOREM 3.6.7. *Let $\mathcal{E}$ be co-Birkhoff and $\Gamma : \mathcal{E} \to \mathcal{E}$ be a covarietor (so that U has a right adjoint H) that preserves regular monos. Then $\mathcal{E}_\Gamma$ is co-Birkhoff.*

It is worth noting that, if $\mathcal{E}$ and Γ satisfy the conditions of Theorem 3.6.7, then each coalgebra $\langle A, \alpha \rangle$ is a regular sub-coalgebra of HA, which is injective if A is. We will use this fact in interpreting the covariety theorem.

In what follows, we assume that $\mathcal{E}$ is co-Birkhoff and Γ preserves regular monos. Under these assumptions, we notice that U preserves and reflects epis, regular monos and colimits (by the dual to Corollary 1.2.15), so that U preserves structure relevant to covarieties.

The regular subcoalgebras of HC play a role in the co-Birkhoff theorem that is analogous to the quotients of FX in Birkhoff's variety theorem. This analogy suggests the following definition.

DEFINITION 3.6.8. A *coequation over C* is a regular subobject of UHC. That is, a coequation K is a(n equivalence class of) regular monos

$$K \rightarrowtail UHC .$$

We take a coequation here to be a regular subobject. This is not the literal dual of sets of equations in $\mathcal{E}^\Gamma$. There, the set of equations E is taken as a binary relation on UFX. So, to dualize a set of equations, one would consider *corelations* on UHC. Since these are less familiar objects, we prefer to consider the regular subcoalgebras themselves, which are dual to the quotients of FX by the pre-congruence generated by E. For a discussion of corelations over coalgebras, see [**Kur00**].

There is also a sense in which the variety theorem is "really" about quotients and orthogonality classes (rather than relations), and this motivates our definition of coequations. Sets of equations arise in the classical setting, but as we saw in Section 3.2.4, the variety theorem can be proved in terms of quotients in a wide variety of categories in which one may not have a natural notion of "equational satisfaction" (apart from orthogonality to a regular epi with regular projective domain, of course).

Some authors (notably, Peter Gumm in [**Gum01a**]) distinguish between single coequations and sets of coequations, a distinction which we do not introduce. In order for this distinction to arise, one states coequation satisfaction in terms of avoidance of certain behaviors. We prefer the more straightforward definition given below (which is equivalent for categories of coalgebras over **Set**), partly because it stresses the coequation-as-predicate view which we exploit later.

Each coequation $K \rightarrowtail UHC$ gives rise to a canonical subcoalgebra, of course, via the $[-]_{HC}$ operator of Section 2.2.2. By Corollary 2.2.9, we know that any homomorphism $f : \langle A, \alpha \rangle \to HC$ factors through $[K]_{HC}$ just in case Uf factors through K. This fact, together with our work on equation satisfaction previously, suggests the following definition.

Definition 3.6.9. Let $K \rightarrowtail HC$ be a coequation over C and $i : [K] \rightarrowtail HC$ the evident inclusion. We say that a coalgebra $\langle A, \alpha \rangle$ forces K (written $\langle A, \alpha \rangle \Vdash_C K$, with the subscript sometimes omitted) just in case $\langle A, \alpha \rangle \perp i$.

In other words, $\langle A, \alpha \rangle \Vdash_C K$ just in case, for every map $p : A \to C$, the adjoint transpose of p factors through K. Intuitively, this means that, no matter how one "colors" the elements of A (with colors chosen from C), each element of A is behaviorally indistinguishable (in the sense of Γ-coalgebras) from an element of K of the same color.

Remark 3.6.10. In some ways, the forcing terminology here does not fit well with the definition, since coequation forcing is really derived from the usual notion of predicate satisfaction (not forcing). Nonetheless, we use the forcing terminology in order to keep the distinction between equation satisfaction (orthogonality to a quotient) and coequation satisfaction (co-orthogonality to a subobject) clear.

Example 3.6.11. Again, let $\Gamma A = X \times A$ and suppose $x \in X$. Let $C = 2$, so HC is $(2 \times X)^\omega$. Let

$$\sigma,\ \tau : \omega \longrightarrow 2 \times X$$

be defined by

$$\sigma(n) = \langle 0,\, x\rangle$$
$$\tau(n) = \langle 1,\, x\rangle$$

for all $n \in \omega$, and let $K = \{\sigma, \tau\}$, so K is a coequation over C.

Consider the coalgebras $\langle A,\, \alpha\rangle$ and $\langle B,\, \beta\rangle$, where $A = \{a\}$ and $\alpha(a) = \langle x,\, a\rangle$ and $B = \{b,\, c\}$ and

$$\beta(b) = \langle x,\, c\rangle$$
$$\beta(c) = \langle x,\, b\rangle$$

Then it is easy to see that $\langle A,\, \alpha\rangle \Vdash K$, while $\langle B,\, \beta\rangle \nVdash K$. Moreover, one can show that $\langle C,\, \gamma\rangle \Vdash K$ just in case, for each $c \in C$, $\gamma(c) = \langle x,\, c\rangle$.

This example should support the view that coequational satisfaction says something about the internal structure of the coalgebra. Bisimilarity tells one about the behavior of a coalgebra — what sorts of output the structure maps can generate. In Section 3.9, we will see that coequation satisfaction is a finer way to distinguish between coalgebras than bisimilarity classes. Example 3.6.11 suggests that coequation satisfaction gives some ability to distinguish the internal composition of distinct coalgebras.

We now have a natural definition for *coequational variety.* A coequational variety is the class of all coalgebras which satisfy some collection of coequations. That is, $\mathbf{V}$ is a coequational covariety iff there is a collection S of regular monics with cofree codomains such that $\mathbf{V} = S_\perp$. With the new terminology, the following theorem is an easy consequence of Theorem 3.6.7.

THEOREM 3.6.12. *Let $\mathcal{E}$ be co-Birkhoff and $\Gamma\colon\mathcal{E}\to\mathcal{E}$ preserve regular monos. Suppose also that $U\colon\mathcal{E}_\Gamma\to\mathcal{E}$ is comonadic (i.e., Γ is a covarietor) and let $\mathbf{V}$ be a full subcategory of $\mathcal{E}_\Gamma$. Then $\mathbf{V}$ is a Birkhoff covariety iff $\mathbf{V}$ is a coequational covariety.*

In Theorem 3.6.7, we showed that any covariety is the co-perp of some collection of regular monos with regular injective codomains. Here, we simply note that each object is a regular subobject of HC for some regular injective C. The dual of this was shown in the proof of Corollary 3.3.2.

EXAMPLE 3.6.13. In [**GS98**], it is shown that the category $\mathbf{Top}^{\mathsf{open}}$ of topological spaces and open maps is a covariety of $\mathbf{Set}_{\mathcal{F}}$ (where $\mathcal{F}$ is the filter functor).

EXAMPLE 3.6.14. Fix a set Z and let $\Gamma\colon\mathbf{Set}\to\mathbf{Set}$ be the functor

$$\Gamma A = Z \times A,$$

so that any Γ-coalgebra $\langle A, \alpha\rangle$ can be viewed as a collection of streams over Z (see Example 1.1.7).

The cofree coalgebra $H\mathbb{N}$ is the final $\mathbb{N}\times Z\times -$ coalgebra — i.e., $H\mathbb{N} = (\mathbb{N}\times Z)^\omega$. Given an element $\sigma \in H\mathbb{N}$, we can define

$$\mathcal{C}ol(\sigma) = \{\pi_1 \circ \sigma(i) \mid i < \omega\}$$

(equivalently, $\mathcal{C}ol(\sigma) = \{\varepsilon_\mathbb{N} \circ t^i(\sigma) \mid i < \omega\}$, where t is the tail destructor). In other words, $\mathcal{C}ol(\sigma)$ is the set of all colors that occur in the stream σ. Define a coequation φ over $\mathbb{N}$ by

$$\varphi = \{\sigma \mid \mathbf{card}(\mathcal{C}ol(\sigma)) < \aleph_0\},$$

(where $\mathbf{card}(X)$ is the cardinality of X) so $\sigma \in \varphi$ just in case only finitely many colors occur in σ.

One can check that, for any Γ-coalgebra $\langle A, \alpha\rangle$, we have $\langle A, \alpha\rangle \Vdash \varphi$ just in case, for all $a \in A$, there is $n \geq 0$, $m > 0$ such that

$$t^n(a) = t^{n+m}(a),$$

(where $\alpha = \langle h, t\rangle$). In other words, $\langle A, \alpha\rangle \Vdash \varphi$ iff each stream in A has only a finite number of "states".

Example 3.6.15. Fix a set $\mathcal{I}$ and let $\Gamma\colon\mathbf{Set}\to\mathbf{Set}$ be the functor

$$\Gamma S = (\mathcal{P}_{\mathsf{fin}} S)^{\mathcal{I}}.$$

In Example 1.1.11, we learned that Γ-coalgebras $\langle S, \sigma\rangle$may be regarded as non-deterministic automata, where the elements of S are the states of the automata. We say that, on input $i \in \mathcal{I}$, there is a transition $s \xrightarrow{i} s'$ just in case $s' \in \sigma(s)(i)$.

The deterministic automata are those automata $\langle S, \sigma\rangle$ such that, for each $s \in S$ and each $i \in \mathcal{I}$, there is at most one s' such that $s \xrightarrow{i} s'$. Let $\mathcal{D}et$ denote the class of deterministic automata, so $\mathcal{D}et \subseteq \mathbf{Set}_\Gamma$. Then it is easy to see that $\mathcal{D}et$ is a covariety in $\mathbf{Set}_\Gamma$.

In fact, one can show that there is a coequation K over 2 colors that defines $\mathcal{D}et$. Namely, define $K \subseteq UH2$ by

$$K = \{x \in UH2 \mid \forall i \in \mathcal{I}\, \forall y, z \in \delta(x)(i)\,.\, \varepsilon_2(y) = \varepsilon_2(z)\},$$

where $\delta\colon UH2 \rightarrowtriangle \Gamma UH2$ is the structure map for $H2$. Then, it is easy to show that

$$\langle A, \alpha\rangle \Vdash K \text{ iff } \langle A, \alpha\rangle \in \mathcal{D}et\,.$$

Example 3.6.16. In this example, we will use an coalgebraic specification syntax for object oriented programming languages adopted from Bart Jacobs (see [**Jac99, RTJ01**], for instance). In Section 2.1.5, we discussed a simple specification in which we gave the signature of certain methods, thus determining a category $\mathbf{Set}_\Gamma$ of models of the class. In this specification, we extend the previous example to include such assertions, thus restricting the coalgebras in which we are interested to a subclass of $\mathbf{Set}_\Gamma$. In fact, these assertions determine a covariety of $\mathbf{Set}_\Gamma$ (and so, in fact, the models form a category of coalgebras for a comonad).

Typically, in coalgebraic specifications, one allows assertions about the "observable" behavior of methods, so that the class of coalgebras which serve as models for the specifications is closed under bisimulations. We extend the usual class of assertions to allow for equations between states of the coalgebras, in order to fully utilize the expressive power of coequations, ignoring the usual conceptual reason for restricting attention to observable behavior.

We consider the class `DecCounter` introduced in Section 2.1.5, although we view it as a basic class (rather than a class which inherits from `Counter`). We add to the previous specification two assertions, as shown here.

begin `DecCounter`

  operations

    $\mathsf{inc}\colon X \to X$

    $\mathsf{val}\colon X \to \mathbb{N}$

    $\mathsf{dec}\colon X \to X + 1$

  assertions

    $\forall x.\, \mathsf{val}(\mathsf{inc}(x)) = \mathsf{val}(x) + 1$

    $\forall x.\, \mathsf{dec}(\mathsf{inc}(x)) \sim x$

    $\forall x.\, \mathsf{dec}(x) = * \text{ iff } \mathsf{val}(x) = 0$

end `DecCounter`

Here, $\sim$ indicates the bisimilarity relation. Thus, the class of models of `DecCounter` is intended to be the class of $- \times \mathbb{N} \times (- + 1)$-coalgebras $\langle A, \alpha\rangle$such that, for every $a \in A$,

$$\begin{aligned} v_\alpha \circ i_\alpha(a) &= v_\alpha(a) + 1, \\ d_\alpha \circ i_\alpha(a) &\sim a, \\ d_\alpha(a) = * &\text{ iff } v_\alpha(a) = 0 \end{aligned}$$

where $\alpha = \langle i_\alpha, v_\alpha, d_\alpha\rangle$. Note that $d_\alpha \circ i_\alpha(a) \sim a$ just in case $! \circ d_\alpha \circ i_\alpha(a) = !(a)$, where $!$ is the unique homomorphism from $\langle A, \alpha\rangle$ to the final $- \times \mathbb{N} \times (- + 1)$-coalgebra.

It is easy to see that the class of models of `DecCounter` is a variety for a coequation K over 1 color (i.e., a subset of the final coalgebra, $H1$). In fact, this is a corollary to the discussion of behavioral covarieties in Section 3.9, but we explicitly give the coequation here. Indeed, let $\langle i_\sigma, v_\sigma, d_\sigma\rangle\colon UH1 \twoheadrightarrow UH1 \times N \times (UH1 + 1)$ be the structure map for the final coalgebra and define sets

$$\begin{aligned} K_1 &= \{x \in UH1 \mid v_\sigma \circ i_\sigma(x) = v_\sigma(x) + 1\} \\ K_2 &= \{x \in UH1 \mid d_\sigma \circ i_\sigma(x) = x\} \\ K_3 &= \{x \in UH1 \mid d_\sigma(x) = * \text{ iff } v_\sigma(x) = 0\} \end{aligned}$$

Let $K = K_1 \cap K_2 \cap K_3$. Let $\langle A,\, \alpha\rangle$ be given, $\alpha = \langle i_\alpha,\, v_\alpha,\, d_\alpha\rangle$. Then, for any $a \in A$,

$$v_\alpha \circ i_\alpha(a) == v_\sigma \circ i_\sigma \circ !(a) \text{ and}$$
$$v_\alpha = v_\sigma \circ !(a) + 1,$$

so $\langle A,\, \alpha\rangle \Vdash K_1$ just in case $\langle A,\, \alpha\rangle$ satisfies the first assertion. Similarly, one checks that $\langle A,\, \alpha\rangle$ forces K_2, K_3 resp., just in case $\langle A,\, \alpha\rangle$ satisfies the second, third resp., assertion.

If, on the other hand, we want the result of applying $\mathsf{dec} \circ \mathsf{inc}$ to return an object to the same state, rather than to a "merely" indistinguishable state[3], then we may replace the second assertion with the related assertion

$$\forall x.\, \mathsf{dec}(\mathsf{inc}(x)) = x.$$

In this case, a coequation over one color does not suffice, again for reasons that we present in Section 3.9. However, there is a related coequation over two colors that defines the class of models for this specification. Namely, we define again three sets as follows:

$$K_1' = \{x \in UH2 \mid v_\sigma \circ i_\sigma(x) = v_\sigma(x) + 1\}$$
$$K_2' = \{x \in UH2 \mid \varepsilon_2 \circ d_\sigma \circ i_\sigma(x) = \varepsilon_2 \circ x\}$$
$$K_3' = \{x \in UH2 \mid d_\sigma(x) = * \text{ iff } v_\sigma(x) = 0\}$$

(We implicitly require that the equation in the definition of set K_2' is well-typed, so that $d_\sigma \circ i_\sigma(x) \in UH2$.) We define $K' \subseteq UH2$ to be the intersection of K_1', K_2' and K_3'. We assert that $\langle A,\, \alpha\rangle \Vdash K'$ just in case K' satisfies the requisite assertions. In particular, $\langle A,\, \alpha\rangle \Vdash K_2'$ just in case, for every $a \in A$,

$$d_\alpha \circ i_\alpha(a) = a. \tag{18}$$

Indeed, suppose that, for every a, Equation (18) holds, and let $p\colon \langle A,\, \alpha\rangle \twoheadrightarrow H2$ be given. Then, for every $a \in A$,

$$\varepsilon_2 \circ d_\sigma \circ i_\sigma \circ p(a) = \varepsilon_2 \circ p \circ d_\alpha \circ i_\alpha(a) = \varepsilon_2 \circ p(a),$$

and so $p(a) \in K_2'$.

On the other hand, suppose that there is an $a \in A$ such that Equation (18) does not hold. If $d_\alpha \circ i_\alpha(a) = *$, then it is easy to show that (for any $p\colon \langle A,\, \alpha\rangle \twoheadrightarrow H2$),

$$p(a) \notin K_2',$$

and so $\langle A,\, \alpha\rangle \nVdash K'$. Suppose, then, that $d_\alpha \circ i_\alpha(a) \in A$. We may define a coloring $\widetilde{p}\colon A \twoheadrightarrow 2$ such that

$$\widetilde{p}(d_\alpha \circ i_\alpha(a)) \neq \widetilde{p}(a).$$

[3]Again, we do not justify this desire here, although one has the idea that the strengthened assertion regarding equality of states is related to assertions describing final methods.

Let $p\colon\langle A,\,\alpha\rangle \to H2$ be the adjoint transpose of $\widetilde{p}$. Then

$$\varepsilon_2 \circ d_\sigma \circ i_\sigma \circ p(a) = \widetilde{p}(d_\alpha \circ i_\alpha(a)) \neq \widetilde{p}(a) = \varepsilon_2(a),$$

and so again $\langle A,\,\alpha\rangle \not\Vdash K'$. Consequently, if $\langle A,\,\alpha\rangle \Vdash K'$, then every $a \in A$ satisfies Equation (18).

3.6.4. Quasi-covarieties of coalgebras. We interpreted the covariety theorem in categories of coalgebras by introducing a notion of coequations. In this section, we revisit the quasi-covariety theorem (Theorem 3.6.3) and interpret it in terms of conditional coequations. Conditional coequations arise as an obvious generalization of coequations, by relaxing the condition that the codomain of the regular mono is cofree. We hope to motivate the use of the term "conditional" by showing that the natural forcing definition for these regular subobjects is equivalent to a conditional forcing of a "proper" coequation.

A similar presentation of conditional coequations (in terms of modal rules) can be found in [**Kur00, Kur99**]). The material of this section covers much of the same ground as Andréyka and Németi covered for the dual (algebraic) theorems in [**Ném82, AN81a, AN81b, AN79b**]. This material was developed independently prior to the author's discovery of the related research.

A coequation over C is just a regular subobject of the cofree coalgebra UHC. More generally, we could consider regular subobjects of the carriers of arbitrary coalgebras. This suggests the following definition, although we postpone justifying the use of the term "conditional".

Definition 3.6.17. A *coequation!conditional* – (over $\langle A,\,\alpha\rangle$) is just a regular subobject

$$K \rightarrowtail A$$

of $A = U\langle A,\,\alpha\rangle$. We sometimes subscript a conditional coequation by α to indicate its codomain.

Recall that $[K]$ is the largest subcoalgebra of $\langle A,\,\alpha\rangle$ whose carrier is contained in K. We say that $\langle B,\,\beta\rangle \Vdash_\alpha K$ just in case $\langle B,\,\beta\rangle \perp i$, where

$$i\colon [K] \rightarrowtail \langle A,\,\alpha\rangle$$

is the inclusion.

Hence, a coequation K over C is just the same as a conditional coequation over HC, and so the conditions $\langle B,\,\beta\rangle \Vdash_C K$ and $\langle B,\,\beta\rangle \Vdash_{HC} K$ are really just the same statement. Nonetheless, we hope no confusion arises from the notational differences between coequations and conditional coequations.

In terms of conditional coequations, Theorem 3.6.3 yields the following theorem.

THEOREM 3.6.18. *Let $\mathcal{E}$ be quasi-co-Birkhoff and Γ preserve regular monos. Then, for any class* $\mathbf{V}$ *of* Γ*-coalgebras,* $\mathbf{V}$ *is a quasi-covariety iff there is a collection* S *of conditional coequations such that*

$$\langle B,\, \beta\rangle \in \mathbf{V} \textit{ iff } \forall K_\alpha \in S \,.\, \langle B,\, \beta\rangle \Vdash_\alpha K$$

EXAMPLE 3.6.19. Let $\mathbb{G} = \langle G,\, \varepsilon,\, \delta\rangle$ be a comonad over $\mathcal{E}$, and assume that G preserves regular monos. Then, the category $\mathcal{E}_{\mathbb{G}}$ of coalgebras for the comonad is a variety in the category $\mathcal{E}_G$ of coalgebras for the endofunctor. Indeed, it is easy to check that $\mathcal{E}_{\mathbb{G}}$ is closed under epis, regular subcoalgebras and coproducts.

If G is not a covarietor[4], then Theorem 3.6.12 does not apply — so, we cannot guarantee a collection of coequations defining $\mathcal{E}_{\mathbb{G}}$. However, we may apply Theorem 3.6.18 in this case, to conclude that there is a collection S of *conditional* coequations defining the covariety $\mathcal{E}_{\mathbb{G}}$.

Indeed, it is not hard to explicitly give a collection S which suffices. For each $\langle A,\, \alpha\rangle \in \mathcal{E}_G$, let Φ_α be the equalizer shown below.

$$\Phi_\alpha \rightarrowtail A \underset{\mathsf{id}_A}{\overset{\varepsilon_A \circ \alpha}{\rightrightarrows}} A$$

That is, Φ_α is just the equalizer of the counit diagram from Definition 2.1.2. Similarly, let Ψ_α be the equalizer of the co-distributivity diagram, shown below.

$$\Psi_\alpha \rightarrowtail A \underset{G\alpha \circ \alpha}{\overset{\delta_A \circ \alpha}{\rightrightarrows}} G^2 A$$

Let S be the collection (abusing set notation)

$$\{\Phi_\alpha \wedge \Psi_\alpha \mid \langle A,\, \alpha\rangle \in \mathcal{E}_G\}.$$

Then it is easy to show that, for any $\langle B,\, \beta\rangle \in \mathcal{E}_G$,

$$\langle B,\, \beta\rangle \in \mathcal{E}_{\mathbb{G}} \text{ iff } \langle B,\, \beta\rangle \Vdash S.$$

In the remainder of this section, we will focus on a special class of quasi-covarieties: those that are defined by a single conditional coequation. Our purpose is to show that the so-called conditional coequations really do reflect a notion of conditional forcing. Namely, given a conditional coequation $i : K \rightarrowtail U\langle A,\, \alpha\rangle$, we may view A as a coequation as well — namely a coequation over the object A of colors (or, if A is not projective, then a coequation over some projective of which A is a regular subobject).

[4]Note: The fact that G is the functor part of a comonad does not seem sufficient to infer that

$$U : \mathcal{E}_G \longrightarrow \mathcal{E}$$

has a right adjoint — although the related forgetful functor

$$U : \mathcal{E}_{\mathbb{G}} \longrightarrow \mathcal{E}$$

certainly does have a right adjoint.

We will show that a coalgebra $\langle B,\, \beta\rangle$forces K just in case $\langle B,\, \beta\rangle \Vdash A \Rightarrow K$ (although we still owe a definition of this latter condition).

Given a pair of coequations K, L over a common object of colors, C, we call $K \Rightarrow L$ an $\Rightarrow$-coequation. We say that a coalgebra $\langle A,\, \alpha\rangle$ *forces* $K \Rightarrow L$ (written $\langle A,\, \alpha\rangle \Vdash_C K \Rightarrow L$) just in case, for every homomorphism $p\colon\langle A,\, \alpha\rangle \twoheadrightarrow HC$, if Up factors through K, then Up factors through L. In other words, $\langle A,\, \alpha\rangle \Vdash_C K \Rightarrow L$ if, whenever $p\colon\langle A,\, \alpha\rangle \twoheadrightarrow HC$ and $\mathsf{Im}\, p \leq [K]$, then $\mathsf{Im}\, p \leq [L]$.

Remark 3.6.20. The condition that

$$\langle A,\, \alpha\rangle \Vdash K \Rightarrow L$$

is not the same as

$$\langle A,\, \alpha\rangle \Vdash K \rightarrow L$$

, where $K \rightarrow L$ is defined in terms of the Heyting algebra structure of $\mathsf{RegSub}(UHC)$. In fact, one can show that, for any K, L, $\langle A,\, \alpha\rangle$, if $\langle A,\, \alpha\rangle \Vdash K \rightarrow L$, then $\langle A,\, \alpha\rangle \Vdash K \Rightarrow L$, but the converse does not hold.

For example, let $\Gamma X = X \times X$. Let

$$\langle \varepsilon_2,\, l,\, r\rangle\colon UH2 \rightarrowtail 2 \times UH2 \times UH2$$

be the structure map for $H2$. Define coequations K, L over 2 by

$$K = \{\sigma \in UH2 \mid \sigma = l(\sigma)\}$$
$$L = \{\sigma \in UH2 \mid \sigma = r(\sigma)\}$$

Let $A = \{a, b\}$ and $\alpha = \langle l_\alpha,\, r_\alpha\rangle\colon A \twoheadrightarrow A \times A$ be defined by

$$\alpha(a) = \langle b,\, b\rangle,$$
$$\alpha(b) = \langle b,\, a\rangle.$$

We will first show that $\langle A,\, \alpha\rangle \Vdash K \Rightarrow L$. Let

$$p\colon A \longrightarrow 2$$

be given such that $\mathsf{Im}\, \widetilde{p} \leq K$, where $\widetilde{p}$ is the adjoint transpose of p. Then, since $\widetilde{p}(a) \in K$, it follows that $\widetilde{p}(a) = \widetilde{p}(b)$. Hence, $\mathsf{Im}\, \widetilde{p} \leq L$.

However, it is not the case that $\langle A,\, \alpha\rangle \Vdash K \rightarrow L$. Let $p(a) = 0$ and $p(b) = 1$. Then, $p(b) \in K$ but $p(b) \notin L$. Hence,

$$\mathsf{Im}\, \widetilde{p} \wedge K \not\leq L$$

and so $\mathsf{Im}\, \widetilde{p} \not\leq K \rightarrow L$.

We wish to show that conditional coequations (in the sense of Definition 3.6.17) coincide with $\Rightarrow$-coequations. More precisely, given any conditional coequation M

over $\langle A, \alpha\rangle$, there is an $\Rightarrow$-coequation $K \to L$ over C, for an appropriate projective object C, such that

$$\langle B, \beta\rangle \Vdash_\alpha M \text{ iff } \langle B, \beta\rangle \Vdash_C K \Rightarrow L.$$

Also, given any $\Rightarrow$-coequation $K \Rightarrow L$, there is an $\langle A, \alpha\rangle$ and $M \leq A$ such that the same equivalence holds. This fact motivates the terminology "conditional coequation", since each conditional coequation can be expressed as a coequation of the form $K \Rightarrow L$.

THEOREM 3.6.21. *Let $\mathcal{E}$ be a Birkhoff category, Γ a varietor the preserves regular monos. Let $\langle A, \alpha\rangle \in \mathcal{E}_\Gamma$ and C an injective object such that $A \leq C$. Then, for any conditional coequation M over $\langle A, \alpha\rangle$, there is an $\Rightarrow$-coequation $K \Rightarrow L$ over C such that*

$$\langle B, \beta\rangle \Vdash_\alpha M \textit{ iff } \langle B, \beta\rangle \Vdash_C K \Rightarrow L. \tag{19}$$

Conversely, if $\mathcal{E}$ has binary intersections, for any $\Rightarrow$-coequation $K \Rightarrow L$ over C, there is a $\langle A, \alpha\rangle$, with $A \leq C$ and a conditional coequation M over $\langle A, \alpha\rangle$ such that (19) holds.

PROOF. Let $m\colon M \rightarrowtail A$, where $A = U\langle A, \alpha\rangle$ and C injective with $i\colon A \rightarrowtail C$. We claim that $\langle B, \beta\rangle \Vdash_\alpha M$ just in case $\langle B, \beta\rangle \Vdash_C A \Rightarrow M$.

Indeed, suppose that $\langle B, \beta\rangle \Vdash M$, so that every homomorphism $\langle B, \beta\rangle \to \langle A, \alpha\rangle$ factors through M. Let $f\colon\langle B, \beta\rangle \to HC$ be given and suppose that f factors through A (as in Figure 9), $f = i \circ g$ for some $g\colon B \to A$. Then, by Corollary 1.2.10, g is also a homomorphism and so g factors through m. Hence, $\langle B, \beta\rangle \Vdash A \Rightarrow M$.

$$\begin{tikzcd}
 & UHC \\
B \arrow[ur, "f"] \arrow[r, "g"] \arrow[dr, "k"'] & A \arrow[u, tail, "i"'] \\
 & M \arrow[u, tail, "m"']
\end{tikzcd}$$

FIGURE 9. $\langle B, \beta\rangle \Vdash_\alpha M$ iff $\langle B, \beta\rangle \Vdash_C A \Rightarrow M$

Conversely, suppose that $\langle B, \beta\rangle \Vdash A \Rightarrow M$ and let $g\colon\langle B, \beta\rangle \to \langle A, \alpha\rangle$ be given. Then $i \circ g$ factors through A and hence factors through M, $i \circ g = i \circ m \circ k$. Since i is monic, we see that g factors through M.

On the other hand, let $K, L \leq UHC$. Let $[K]_{HC} = \langle A, \alpha\rangle$ (so $\langle A, \alpha\rangle$ is the largest subcoalgebra of HC contained in K). We claim that $\langle B, \beta\rangle \Vdash_\alpha L \cap A$ just in case $\langle B, \beta\rangle \Vdash_C K \Rightarrow L$.

Suppose that $\langle B, \beta\rangle \Vdash_\alpha L \cap A$. Let $f:\langle B, \beta\rangle \twoheadrightarrow HC$ be given and suppose that $\mathsf{Im}\, Uf \leq K$. Then, by Corollary 2.2.9, $\mathsf{Im}\, f \leq [K] = \langle A, \alpha\rangle$ and so $\mathsf{Im}\, Uf \leq L \cap A \leq L$. Hence, $\langle B, \beta\rangle \Vdash_C K \Rightarrow L$.

Suppose now that $\langle B, \beta\rangle \Vdash_C K \Rightarrow L$, and let $g:\langle B, \beta\rangle \twoheadrightarrow \langle A, \alpha\rangle$ be given. Since $A \leq K$, the map g factors through L and hence through $A \cap L$. Thus,

$$\langle B, \beta\rangle \Vdash_\alpha A \cap L.$$

$\square$

3.7. Uniformly co-Birkhoff categories

In Section 3.4, we considered those categories in which every variety $\mathbf{V}$ is of the form $\mathbf{V} = \{p: A \twoheadrightarrow \bullet\}_\perp$ for some regular epi p, regular projective A. In the case of categories of algebras, we could take A to be free over some projective object X "of variables". Thus, if $\mathcal{E}^\Gamma$ is uniformly Birkhoff, then there is some X such that every variety is definable by a set of equations over X. In the classical setting of $\mathbf{Set}^{\mathbb{P}}$, where $\mathbb{P}$ is a polynomial functor, we could take X to be any infinite set, in accordance with the 1935 Birkhoff variety theorem.

We wish to consider the analogous conditions for categories of coalgebras and their covarieties. Namely, what conditions suffice to conclude that there is an regular injective C such that every covariety $\mathbf{V}$ of $\mathcal{E}_\Gamma$ is definable by a coequation over C? In more detail, we want conditions that ensure that, for every $\mathbf{V}$, there is a $K \rightarrowtail UHC$ such that

$$\mathbf{V} = \{\langle A, \alpha\rangle \in \mathcal{E}_\Gamma \mid \langle A, \alpha\rangle \Vdash_C K\}.$$

As we shall see in Section 3.8, K can always be taken to be the carrier of a subcoalgebra. Thus, the question is when $\mathbf{V} = \{i: \bullet \rightarrowtail HC\}_\perp$ for some regular mono Γ-homomorphism i.

Definition 3.7.1. A co-Birkhoff category $\mathcal{C}$ is *uniformly co-Birkhoff* just in case there is an regular injective $C \in \mathcal{C}$ such that, for every variety $\mathbf{V}$, there is a regular mono $i: K \rightarrowtail C$ such that $\mathbf{V} = \{i\}_\perp$. In this case, we call C a *coequational codomain.*

3.7.1. Conjunctly irreducible coalgebras and conjunct sums. Here, we prove an analogue to Birkhoff's subdirect representation theorem ([**Bir44**]). The so-called conjunct representation theorem was first proved by Gumm and Schröder for categories of coalgebras over **Set** (see [**Gum01b, Gum98, Gum99**]). We generalize their work here. We begin by stating the relevant definitions for the classical theorem as well as the theorem itself, which we take from [**Grä68**]. For the classical theorem, we work in $\mathbf{Set}^{\mathbb{P}}$ for a polynomial functor $\mathbb{P}$.

Call an algebra $\langle A, \alpha\rangle$ *subdirectly irreducible* just in case, whenever $\{\Theta_i\}_{i\in I}$ is a family of congruences on $\langle A, \alpha\rangle$ with $\bigwedge \Theta_i = \Delta_\alpha$, the diagonal on $\langle A, \alpha\rangle$, then one of the Θ_i equals Δ_α.

THEOREM 3.7.2 (Subdirect representation). *For any $\langle A, \alpha\rangle$ in $\mathbf{Set}^{\mathbb{P}}$, there is a family $\{\langle A_i, \alpha_i\rangle\}_{i\in I}$ such that each $\langle A_i, \alpha_i\rangle$ is a quotient of $\langle A, \alpha\rangle$ and*

$$\langle A, \alpha\rangle \leq \prod \langle A_i, \alpha_i\rangle.$$

The material that follows is a good example of the limitations of formal dualities. The proof of the conjunct representability theorem given here does not follow from a simple dualization of the proof of the subdirect representability theorem. The classical theorem relies on finding, for each $a \neq b$ in an algebra $\langle A, \alpha\rangle$, a congruence $R_{a,b}$ such that $\neg R_{a,b}(a, b)$ and taking the product of the $A/R_{a,b}$ for pairs of distinct a, b. This approach does not easily dualize to yield the coalgebraic theorem, and so a different approach is used. Nonetheless, an analogous result is obtained — the conclusion of the theorem is dual to the subdirect representation theorem, but the assumptions required and the methods used are not dual.

Throughout this section, we assume that $\mathcal{E}$ has epi-regular mono factorizations, finite limits and all coproducts and that Γ preserves regular monos.

The following definition is dual to subdirect products in the classical theorem.

DEFINITION 3.7.3. Let $A \in \mathcal{C}$. A *conjunct covering of A* is a collection of regular monos

$$\{\, C_i \overset{c_i}{\rightarrowtail} A \mid i \in I\}$$

such that the map

$$\textstyle\coprod_{i\in I} C_i \xrightarrow{[c_i]_{i\in I}} A$$

is a regular epi.

The requirement that $\coprod C_i \to A$ is a regular epi, instead of just epi, is necessary only so that conjunct covers are stable under pullbacks. In what follows, we may replace regular epis with epis if we also require that all epis are stable under pullback, rather than just requiring that $\mathcal{C}$ is a regular category.

Next, we dualize the notions of subdirect irreducibility and representability.

DEFINITION 3.7.4. We say that an object $A \in \mathcal{C}$ is *conjunctly irreducible* iff, whenever we have a non-empty conjunct covering of A,

$$\textstyle\coprod_{i\in I} C_i \xrightarrow{[c_i]_{i\in I}} A\ ,$$

then one of the c_i's is an epi (and hence an isomorphism).

Definition 3.7.5. We say that a conjunct covering

$$\{c_i : C_i \rightarrowtail A\}_{i\in I}$$

of A is a *conjunct representation of A* if each C_i is conjunctly irreducible. We say that $\mathcal{C}$ is *conjunctly presentable* if each object $A \in \mathcal{C}$ has a conjunct representation.

An object A is, by definition, conjunctly irreducible just in case, for every family $\{C_i\}_{i\in I}$ of regular subobjects of A, if $\coprod C_i$ covers A, then one of the C_i covers A. The following theorem shows that whenever A is conjunctly irreducible, then, for any family $\{C_i\}_{i\in I}$ which covers A, one of the $C_i \to A$ is epi.

Theorem 3.7.6. *Let $\mathcal{C}$ be a category with epi-regular mono factorizations. An object $A \in \mathcal{C}$ is conjunctly irreducible iff for every collection of maps (not necessarily monic)*

$$\{\ C_i \xrightarrow{c_i} A \mid i \in I\}$$

such that the induced map

$$\textstyle\coprod_{i\in I} C_i \xrightarrow{[c_i]_{i\in I}} A$$

is a regular epi, there is some $i \in I$ such that c_i is epi.

Proof. Suppose that A is conjunctly irreducible. For each i, take the epi-regular mono factorization, as shown below.

$$\begin{tikzcd}
C_i \arrow[rr, bend left, "c_i"] \arrow[dr, "p_i"'] & & A \\
& D_i \arrow[ur, tail, "d_i"'] &
\end{tikzcd}$$

Then, it is easy to see that the d_i's form a regular cover of A, and so there is some i such that d_i is an epi. Hence, c_i is also an epi. □

For the remainder of this section, we will be interested in categories in which coproducts commute with pullback, a generalization of distributive categories. Such categories are called *extensive*. We present the definition here and state without proof a theorem (found as Theorem 5.5.8 in [**Tay99**]) giving an equivalent definition of extensive. See also [**Coc93**] for a discussion of extensive categories, a subject we return to in Section 4.1, where we show that $\mathcal{E}_\Gamma$ is extensive, given that $\mathcal{E}$ is extensive and Γ preserves regular monos and pullbacks along regular monos.

Definition 3.7.7. A category with finite coproducts is *extensive* if, in the diagram in Figure 10, the squares are pullbacks just in case the top row is a coproduct diagram.

Any extensive category with finite limits is distributive.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z \arrow[d] & Y \arrow[l] \arrow[d] \\ A \arrow[r, "\kappa_A"'] & A+B & B \arrow[l, "\kappa_B"] \end{tikzcd}$$

FIGURE 10. In an extensive category, the squares are pullbacks iff $Z = X + Y$.

THEOREM 3.7.8. *Let $\mathcal{C}$ have all pullbacks. $\mathcal{C}$ is extensive just in case*

- *$\mathcal{C}$ has a strict initial object, i.e., any map $\bullet \to 0$ is an isomorphism;*
- *coproducts are disjoint, i.e., the diagram below is always a pullback;*

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\ B \arrow[r] & A+B \end{tikzcd}$$

- *coproducts are stable under pullback, i.e., if the squares in Figure 10 are pullbacks, then $Z \cong X + Y$.*

In particular, then, every topos is extensive.

We use the property that coproducts commute with pullbacks to ensure that the pullback of a conjunct covering is again a conjunct covering. This, in turn, will ensure that a coalgebra is conjunctly irreducible just in case it is generated by a conjunctly irreducible subobject.

THEOREM 3.7.9. *Assume $\mathcal{C}$ is a regular, extensive category. Let $f : A \twoheadrightarrow B$ be given and let*

$$\{\, C_i \overset{c_i}{\rightarrowtail} B \mid i \in I\}$$

be a conjunct covering of B. Then pulling each C_i back along f yields a conjunct covering of A. In other words, pullbacks preserve conjunct coverings.

PROOF. For each i, take the pullback

$$\begin{tikzcd} D_i \arrow[r, rightarrowtail, "d_i"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ C_i \arrow[r, rightarrowtail, "c_i"'] & B \end{tikzcd} \tag{20}$$

Pullbacks preserve regular monos, so all that remains is to show that the map from the coproduct of the D_i's is also a regular epi. But, since pullbacks commute with coproducts, the following square is a pullback.

$$\begin{tikzcd} \coprod_i D_i \arrow[r, "{[d_i]}"] \arrow[d] & A \arrow[d, "f"] \\ \coprod_i C_i \arrow[r, "{[c_i]}"'] & B \end{tikzcd}$$

By regularity, the top arrow is a regular epi. □

THEOREM 3.7.10. *Let $\mathcal{C}$ be regular and extensive. Let A be conjunctly irreducible and suppose $f : A \twoheadrightarrow B$ is an epi. Then B is conjunctly irreducible.*

PROOF. Let

$$\{ C_i \overset{c_i}{\rightarrowtail} B \mid i \in I\}$$

be a regular covering of B and take the pullbacks as in the proof of Theorem 3.7.9. Since A is conjunctly irreducible, there is an i such that $D_i \cong A$. Because of the commutativity of the pullback square (20), c_i is an epi and hence is an isomorphism. □

DEFINITION 3.7.11. Let $\langle A, \alpha\rangle$ be a Γ-coalgebra and S a regular subobject of A. We say that S *generates* $\langle A, \alpha\rangle$ if no proper regular subcoalgebra of $\langle A, \alpha\rangle$ contains S.

Recall that in Section 2.3, we showed that whenever Γ preserves non-empty intersections, the subcoalgebra forgetful functor $U_\alpha : \mathsf{SubCoalg}\,\langle A, \alpha\rangle \to \mathsf{RegSub}\, A$ has a left adjoint, $\langle\rangle_\alpha : \mathsf{RegSub}(A) \to \mathsf{SubCoalg}\,\langle A, \alpha\rangle$. In this case, it is easy to see that S generates $\langle A, \alpha\rangle$ just in case $\langle S\rangle_\alpha = \langle A, \alpha\rangle$.

THEOREM 3.7.12. *Let $\mathcal{E}$ be a regular, extensive category. Suppose further that $\mathcal{E}$ is conjunctly presentable. Let $\langle A, \alpha\rangle$ be a Γ-coalgebra. Then $\langle A, \alpha\rangle$ is conjunctly irreducible (in $\mathcal{E}_\Gamma$) iff there is some regular subobject S of A such that S is conjunctly irreducible (in $\mathcal{E}$) and S generates $\langle A, \alpha\rangle$.*

PROOF. Suppose that there is some conjunctly irreducible S which generates $\langle A, \alpha\rangle$ and let

$$\{ \langle C_i, \gamma_i\rangle \overset{c_i}{\rightarrowtail} \langle A, \alpha\rangle \mid i \in I\}$$

be a conjunct covering of $\langle A, \alpha\rangle$. Then

$$\{ C_i \overset{c_i}{\rightarrowtail} A \mid i \in I\}$$

is a conjunct covering of A in $\mathcal{E}$ (since U preserves regular monos and regular epis). Consequently, when we pull it back to S, as shown below, we have a conjunct covering of S.

$$\begin{tikzcd} \coprod_i C_i' \arrow[r] \arrow[d, tail] \arrow[dr, phantom, very near start, "\lrcorner"] & S \arrow[d, tail] \\ \coprod_i C_i \arrow[r] & A \end{tikzcd}$$

Hence, for some i, C_i' is isomorphic to S, and so we have $S \leq C_i$ and so, since S generates $\langle A, \alpha\rangle$, $\langle A, \alpha\rangle \cong \langle C_i, \gamma_i\rangle$.

For the converse, assume that, for every conjunctly irreducible C in A, there is a proper regular subcoalgebra $\langle D, \delta\rangle$ containing C. Pick a conjunct representation of A,

$$\{\, C_i \overset{c_i}{\rightarrowtail} A \mid i \in I\}.$$

For each C_i, pick a proper regular subcoalgebra $\langle D_i, \delta_i\rangle$ of $\langle A, \alpha\rangle$ that contains C_i. Then the $\langle D_i, \delta_i\rangle$'s form a conjunct cover of $\langle A, \alpha\rangle$ and none of the $\langle D_i, \delta_i\rangle$'s are isomorphic to $\langle A, \alpha\rangle$. Consequently, $\langle A, \alpha\rangle$ is not conjunctly irreducible. □

Hence, if Γ preserves non-empty intersections, so that we have $\langle\rangle \dashv U$, the category $\mathcal{E}_\Gamma$ is conjunctly presentable if $\mathcal{E}$ is. This is because every coalgebra $\langle A, \alpha\rangle$can be conjunctly covered by $\langle C_i\rangle$, where $\{C_i\}_{i\in I}$ is a conjunct representation of A.

Theorem 3.7.13. *Let $\mathcal{E}$ be regular, extensive and conjunctly presentable and let Γ preserve non-empty intersections. Then $\mathcal{E}_\Gamma$ is conjunctly presentable.*

Proof. Let $\langle A, \alpha\rangle$ be a Γ-coalgebra and let

$$\{c_i : C_i \rightarrowtail A\}$$

be a conjunct representation of A (in $\mathcal{E}$). Then, by Theorem 3.7.12, each $\langle C_i\rangle_\alpha$ is conjunctly irreducible. Because the diagram below commutes (in $\mathcal{E}$), and because U creates colimits, the $\langle C_i\rangle$'s form a conjunct representation of $\langle A, \alpha\rangle$.

$$\begin{tikzcd} \coprod U\langle C_i\rangle \arrow[r, tail] & A \\ \coprod C_i \arrow[u] \arrow[ur] & \end{tikzcd}$$

□

In the remainder of this section, we give sufficient conditions to ensure that the base category $\mathcal{E}$ is conjunctly presentable. As a corollary to this, **Set** is a conjunctly presentable, but that fact is easy enough that one would prove it directly. We offer this discussion to indicate that other categories are also presentable — although the assumption we use (that $\mathcal{E}$ is well-pointed) is a strong assumption. In a sense, we show here that sufficiently "**Set**-like" categories are conjunctly presentable.

Definition 3.7.14. Let $\mathcal{C}$ be a cocomplete category. A set $\{U_i\}_{i\in I}$ of objects of $\mathcal{C}$ is a *(regular, resp.) generating family* if, for every $C \in \mathcal{C}$, there is a $J \subset I$ such that there is (regular, resp.) epi

$$\coprod_{i\in J} U_i \twoheadrightarrow C\,.$$

It is immediate that, if $\mathcal{E}$ has a regular generating family of conjunctly irreducible objects, then $\mathcal{E}$ is conjunctly presentable. It is almost as obvious that, if $\mathcal{E}$ has a regular generating family in which each object is conjunctly presentable, then $\mathcal{E}$ is conjunctly presentable, as the following theorem shows.

Theorem 3.7.15. *Let $\mathcal{E}$ have a set $\{U_i\}_{i\in I}$ of regular generators and assume that each U_i has a conjunct representation. Then $\mathcal{E}$ is conjunctly presentable.*

Proof. Let S_i be the set of conjunctly irreducible objects in the conjunct representation of U_i. Then $\bigcup_{i\in I} S_i$ is a regular generating family of conjunctly irreducible objects. □

The following definition is stated in terms of toposes with all colimits just for consistency with the definition of generating families above. Both definitions are more general than we have stated here, but we are at present interested only in generating families in cocomplete categories, as it is in these categories that the concept is closely related to conjunct coverings.

Definition 3.7.16. A topos $\mathcal{E}$ with all colimits (equivalently, limits) is *well-pointed* if $0 \neq 1$ and 1 is a (regular) generator for $\mathcal{E}$ (i.e., $\{1\}$ is a generating family).

See any good topos theory book for a discussion of well-pointed toposes, including [**LM92, BW85**] or [**Bor94**, Volume III]. In what follows, we use the fact that, in a topos, every epi is regular.

Claim 3.7.17. *In a well-pointed topos, any $A \neq 0$ has a global point.*

Proof. If $A \neq 0$, then the classifying maps for $0 \leq A$ and $A \leq A$ are distinct. Hence, there must be a map $1 \to A$ which distinguishes them. □

Theorem 3.7.18. *Let $\mathcal{E}$ be a well-pointed topos with all colimits and suppose that $\mathcal{E}$ is regular. Then $\mathcal{E}$ is conjunctly presentable.*

Proof. We first show that 1 is conjunctly irreducible. Let

$$\{c_i : C_i \rhd\!\!\longrightarrow 1\}$$

be a conjunct covering. Then $\coprod C_i \neq 0$ and so, for some i, $C_i \neq 0$. Hence, C_i has a global point x. Since the composite

$$1 \xrightarrow{x} C \overset{c_i}{\rhd\!\!\longrightarrow} 1$$

is the identity, c_i is an isomorphism.

Since every object is covered by a coproduct of 1's, the result follows. $\square$

As a final aside, we point out that in a well-pointed topos with all colimits, the only conjunctly irreducible objects are 0 and 1.

COROLLARY 3.7.19. *An object C in a well-pointed topos with all colimits is conjunctly irreducible iff $C \cong 1$ or $C \cong 0$.*

PROOF. Clearly, 0 and 1 are irreducible. Let C be given, and suppose C is conjunctly irreducible. Suppose also that C is not initial. Then C can be written as a non-empty coproduct of 1's. Hence, since C is assumed irreducible, C must be isomorphic to 1. $\square$

3.7.2. Bounded functors. Throughout this section, we assume that $\mathcal{E}$ is co-Birkhoff and that Γ is a covarietor that preserves regular monos. We use some of the terminology of the preceding section in order to define the relevant terms here — in particular, in order to define *bounded functor*. It should be noted that Jiri Adámek recently showed that bounded **Set** functors are just the accessible functors ([**AP01**]), although we do not exploit this discovery in what follows.

DEFINITION 3.7.20. Let $\mathcal{E}$ be conjunctly presentable and let $\Gamma : \mathcal{E} \to \mathcal{E}$. We say that Γ is a *bounded by $C \in \mathcal{E}$* just in case for each Γ-coalgebra $\langle A, \alpha \rangle$, there is a conjunct representation

$$\{c_i : C_i \rightarrowtail A\}_{i \in I}$$

of A such that, for each i, there is a regular subcoalgebra

$$\langle D_i, \delta_i \rangle \leq \langle A, \alpha \rangle$$

such that $C_i \leq D_i \leq C$.

The lemma below is the dual of Lemma 3.4.12.

LEMMA 3.7.21. *Let $\mathbf{V}$ be a covariety in the co-Birkhoff category $\mathcal{C}$ and let A be a regular subobject of B. Then*

$$\{\varepsilon_B^{\mathbf{V}}\}_\perp \subseteq \{\varepsilon_A^{\mathbf{V}}\}_\perp.$$

PROOF. Let $C \perp \varepsilon_B^{\mathbf{V}}$ and let $f : C \to A$ be given. Let

$$A \rightarrowtail B \rightrightarrows E$$

be an equalizer diagram. Because $G^{\mathbf{V}}$ preserves equalizer diagrams (the dual of Corollary 3.2.14), the bottom row in Figure 11 is an equalizer. The vertical arrows are counits for the comonad $\langle G^{\mathbf{V}}, \varepsilon^{\mathbf{V}}, \delta^{\mathbf{V}} \rangle$. Because $C \perp \varepsilon_B^{\mathbf{V}}$, we have a map $C \to G^{\mathbf{V}} B$,

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[dr, dotted] \arrow[drr, dotted] & A \arrow[r, tail] & B \arrow[r, shift left] \arrow[r, shift right] & E \\
& G^{\mathbf{V}}A \arrow[u, tail] \arrow[r, tail] & G^{\mathbf{V}}B \arrow[u, tail] \arrow[r, shift left] \arrow[r, shift right] & G^{\mathbf{V}}E \arrow[u, tail]
\end{tikzcd}$$

FIGURE 11. If $A \le B$ then $\{\varepsilon^{\mathbf{V}}_B\}_\perp \subseteq \{\varepsilon^{\mathbf{V}}_A\}_\perp$.

as shown. By naturality, the diagram

$$\begin{tikzcd}
C \arrow[r, tail] & G^{\mathbf{V}}B \arrow[r, shift left] \arrow[r, shift right] & G^{\mathbf{V}}E
\end{tikzcd}$$

commutes, yielding the factorization of f through $G^{\mathbf{V}}A$, as shown. $\square$

The next theorem is another example of a "dual" theorem which is not proved "by duality". The algebraic analogue to Theorem 3.7.22 is Theorem 3.4.15, in which we showed that, if $\mathcal{E}$ is Birkhoff with enough retractably presentable regular projectives and a regular projective X satisfying certain properties and Γ preserves filtered colimits and regular epis, then $\mathcal{E}^\Gamma$ is uniformly Birkhoff. These assumptions do not dualize in a reasonable way (since the dual of the conditions on X, say, involves dualizing finitely presentable objects, and the result of that is unfamiliar and apparently uncommon). Hence, we offer a separate proof of the uniformly co-Birkhoff theorem here, one which is apparently simpler than the algebraic version, but again does not dualize in an obvious way.

The reader should note that in the following theorem, we do not suppose that $\mathcal{E}$ is uniformly co-Birkhoff. Again, this is a different approach to reach a result analogous to that of Theorem 3.4.15.

THEOREM 3.7.22. *Let $\mathcal{E}$ be conjunctly presentable. If $\Gamma : \mathcal{E} \to \mathcal{E}$ is bounded by C, then $\mathcal{E}_\Gamma$ is uniformly co-Birkhoff.*

PROOF. Let D be a regular injective object of $\mathcal{E}$ such that $C \le D$ ($\mathcal{E}$ has enough regular injectives). Clearly Γ is bounded by D. We already know $\mathbf{V} \subseteq \{\varepsilon^{\mathbf{V}}_{HD}\}_\perp$. We will prove the other inclusion, in order to conclude that HD is a coequational codomain.

Let $\langle A,\, \alpha\rangle \perp \varepsilon^{\mathbf{V}}_{HD}$ and let

$$\{c_i : C_i \rightarrowtail A\}$$

be a conjunct representation of A. For each i, choose a regular subcoalgebra $\langle D_i,\, \delta_i\rangle$ of A such that $C_i \le D_i \le D$. Because $\{\varepsilon^{\mathbf{V}}_{HD}\}_\perp$ is closed under regular subobjects, we see that $\langle D_i,\, \delta_i\rangle \perp \varepsilon^{\mathbf{V}}_{HD}$ for each $\langle D_i,\, \delta_i\rangle$.

Each $U\eta_{\delta_i}$ is a regular mono (since $\varepsilon_U \circ U\eta = 1$), and U reflects regular monos. Hence, since U and H also preserve regular monos, we see that

$$\langle D_i,\, \delta_i\rangle \leq HD_i \leq HD.$$

We apply Lemma 3.7.21 to conclude that each $\langle D_i,\, \delta_i\rangle \perp \varepsilon^{\mathbf{V}}_{\delta_i}$. Hence, each $\langle D_i,\, \delta_i\rangle \in \mathbf{V}$ (Corollary 3.6.5), and so $\coprod \langle D_i,\, \delta_i\rangle \in \mathbf{V}$. Because U creates coproducts and reflects epis,

$$\coprod \langle D_i,\, \delta_i\rangle \twoheadrightarrow \langle A,\, \alpha\rangle$$

is an epi, and so $\langle A,\, \alpha\rangle \in \mathbf{V}$. $\square$

3.8. Invariant coequations

In Section 3.5, we presented Birkhoff's deductive completeness theorem in terms of closure operators on sets of equations, that is, subsets of $UFX \times UFX$. In this section, we present the dual theorem, which we call the invariance theorem. As we will see, the dual of deductive closure leads to two **S4** necessity operators for coequations. The first operator is just the subcoalgebra operator $\Box$ from Section 2.2.2. The second operator, $\boxtimes$, was first introduced in [**Hug01**], in which this material is also covered. It takes a coequation to the largest *endomorphism-invariant* sub-coequation.

Throughout this section, we assume that $\mathcal{E}$ is a co-Birkhoff category and Γ is a regular-mono-preserving covarietor, so that the coalgebraic covariety theorem applies.

The deductive completeness theorem says that a set E of equations is the equational theory for some class $\mathbf{V}$ of algebras just in case E is deductively closed. Previously, we introduced a closure operator

$$\mathcal{I}d_X : \mathsf{Sub}(\mathcal{E}^\Gamma) \longrightarrow \mathsf{Rel}(UFX, UFX),$$

taking a class $\mathbf{V}$ to the largest set of equations over X which $\mathbf{V}$ satisfies. This operator forms a Galois correspondence with the operator

$$\mathcal{S}at_X : \mathsf{Rel}(UFX, UFX) \longrightarrow \mathsf{Sub}(\mathcal{E}^\Gamma),$$

taking a set of equations to the variety it defines. Dually, we may define operators, for each injective C,

$$\mathcal{C}o\mathcal{I}d_C : \mathsf{Sub}(\mathcal{E}_\Gamma) \longrightarrow \mathsf{RegSub}(UHC),$$
$$\mathcal{F}rc_C : \mathsf{RegSub}(UHC) \longrightarrow \mathsf{Sub}(\mathcal{E}_\Gamma),$$

in the obvious way. Namely, if $\mathbf{V}$ is a class of coalgebras, and $K \leq UHC$, then (abusing set notation in the second definition)

$$\mathcal{C}o\mathcal{I}d(\mathbf{V}) = \bigwedge\{L \leq UHC \mid \mathbf{V} \Vdash L\},$$
$$\mathcal{F}rc(K) = \{\langle A,\, \alpha\rangle \mid \langle A,\, \alpha\rangle \Vdash K\}.$$

Again, of course, we have a pair of adjoint functors, $\mathcal{C}o\mathcal{I}d \dashv \mathcal{F}rc$. Note, however, that both of these functors are covariant. That is,

$$\mathcal{C}o\mathcal{I}d(\mathbf{V}) \leq K \text{ iff } \mathbf{V} \leq \mathcal{F}rc(K).$$

In ibid, the coequation $\mathcal{C}o\mathcal{I}d(\mathbf{V})$ is called a *generating coequation* , since it gives a measure of the "coequational commitment" of the class $\mathbf{V}$. In particular, whenever $\mathbf{V} \Vdash K$, then $\mathcal{C}o\mathcal{I}d(\mathbf{V}) \leq K$.

Our goal, then, is to find conditions on $K \leq UHC$ such that $K = \mathcal{C}o\mathcal{I}d(\mathbf{V})$ for some class $\mathbf{V}$ of coalgebras. As before, the conditions should be "syntactic" terms, without reference to the coalgebras which force K. We begin by introducing the notion of endomorphism-invariance.

The notion of an endomorphism-invariant coequation arises as the dual to a stable set of equations, that is, a set of equations closed under substitutions of terms for variables. More accurately, endomorphism-invariant coequations are dual to stable quotients, and the invariance operator plays the role of closure under substitution.

DEFINITION 3.8.1. A coequation K over C is *endomorphism-invariant* just in case, for every homomorphism

$$p\colon HC \longrightarrow HC,$$

the image of K under p is contained in K, i.e.,

$$\exists_p K \leq K.$$

Sometimes, we write *endo-invariant*, or possibly just *invariant*, for endomorphism-invariant. However, the reader should be aware that other authors say that a subobject K of $U\langle A,\, \alpha\rangle$ is invariant (or "α-invariant") just in case K is the carrier of a subcoalgebra of $\langle A,\, \alpha\rangle$. Indeed, we use a similar terminology ("α, β-invariant") in the proof of Theorem 2.5.25. This is a different concept than endomorphism-invariant.

An endomorphism $HC \to HC$ is equivalent to a "re-painting" of the elements of UHC, again drawing colors from C. A coequation K is endo-invariant if, under every such re-painting, the elements of K are behaviorally (including color) indistinguishable from elements that were already in K before the re-painting.

REMARK 3.8.2. The definitions of *endomorphism-invariant*, $\mathcal{C}o\mathcal{I}d$ and $\mathcal{F}rc$ could be stated for arbitrary coalgebras, rather than just cofree coalgebras. We ignore the generality here in favor of focusing on the problem at hand, but see [**Hug01**].

REMARK 3.8.3. Coequations over 1 are always endomorphism invariant, and so $\boxtimes_1$ is just the identity.

Let $\mathsf{Inv}(C)$ denote the full subcategory of $\mathsf{RegSub}(UHC)$ consisting of the invariant coequations over C and let

$$I_C : \mathsf{Inv}(C) \longrightarrow \mathsf{RegSub}(UHC)$$

be the inclusion functor.

THEOREM 3.8.4. *I_C has a right adjoint.*

PROOF. Let $K \leq UHC$ and define

$$\mathfrak{P}_K = \{L \leq UHC \mid \forall p : HC \longrightarrow HC \, (\exists_p L \leq K)\}.$$

We define a functor $J_C : \mathsf{RegSub}(UHC) \to \mathsf{RegSub}(UHC)$ by

$$J_C(K) = \bigvee_{L \in \mathfrak{P}_K} L,$$

omitting the subscripts on I and J when convenient.

We first show that JK is invariant. Let

$$r : HC \longrightarrow HC$$

be given. In order to show that $\exists_r JK \leq JK$, it suffices to show that $\exists_r JK \in \mathfrak{P}_K$, i.e., for every homomorphism $p : HC \to HC$, we have $\exists_p(\exists_r JK) \leq K$. A quick calculation shows

$$\exists_p \exists_r JK = \exists_{p \circ r} \bigvee_{L \in \mathfrak{P}_K} L = \bigvee_{L \in \mathfrak{P}_K} \exists_{p \circ r} L \leq K.$$

Next, we show that $I \dashv J$. Let L be invariant. If $L \leq K$, then, for every endomorphism p,

$$\exists_p L \leq L \leq K,$$

so $L \in \mathfrak{P}_K$ and hence $L \leq JK$. On the other hand, if $L \leq JK$, then

$$L \leq JK \leq K.$$

$\square$

The adjoint pair $I_C \dashv J_C$ yields a comonad $\boxtimes_C = I_C J_C$. We will prove that $\boxtimes_C$ is an **S4** necessity operator, just as we showed in Theorem 2.2.16 that $\Box$ is an **S4** operator. First, some examples calculating $\boxtimes K$ for a coequation K.

EXAMPLE 3.8.5. Let $\Gamma S = (\mathcal{P}_{\mathsf{fin}} S)^{\mathcal{I}}$, as in Example 3.6.15. Recall that the class of deterministic automata $\mathcal{D}et$ forms a covariety of $\mathbf{Set}_\Gamma$, where the defining coequation K over 2 is given by

$$K = \{x \in UH2 \mid \forall i \in \mathcal{I} \, \forall y, z \in \sigma(x)(i) \, . \, \varepsilon_2(y) = \varepsilon_2(z)\}.$$

It is easy to show that

$$\boxtimes K = \{x \in UH2 \mid \forall i \in \mathcal{I} \, \forall y, z \in \sigma(x)(i) \, . \, y = z\},$$

or, more simply,

$$\boxtimes K = \{x \in UH2 \mid \forall i \in \mathcal{I}\,.\ \mathbf{card}(\sigma(x)(i)) < 2\}.$$

EXAMPLE 3.8.6. Recall the functor $\Gamma A = Z \times A$ and the coequation φ over $\mathbb{N}$ defined by

$$\varphi = \{\sigma \mid \mathbf{card}(\mathcal{C}ol(\sigma)) < \aleph_0\},$$

from Example 3.6.14. For each $\sigma \in UH\mathbb{N}$, let

$$\mathcal{S}t(\sigma) = \{t^n(\sigma) \mid n \in \omega\},$$

where $\langle \varepsilon_{\mathbb{N}},\, h,\, t\rangle : UH\mathbb{N} \rhd\!\!\rhd \mathbb{N} \times Z \times UH\mathbb{N}$ is the structure map for $H\mathbb{N}$. Then

$$\boxtimes\varphi = \{\sigma \mid \mathbf{card}(\mathcal{S}t(\sigma)) < \aleph_0\}.$$

In other words, $\sigma \in \boxtimes\varphi$ just in case the coalgebra $[\sigma] \leq H\mathbb{N}$ generated by σ, as in Section 2.3, forces φ.

THEOREM 3.8.7. *$\boxtimes$ is an* **S4** *necessity operator.*

PROOF. Again, since $\boxtimes$ is a comonad, it suffices to show that $\boxtimes$ preserves finite limits. It is obvious that $\top$ is invariant (so $\top \leq \boxtimes\top$). We now show that

$$\boxtimes K \wedge \boxtimes L \leq \boxtimes(K \wedge L).$$

Let $p: HC \twoheadrightarrow HC$ be given (where K, L are coequations over C). Then

$$\exists_p(\boxtimes K \wedge \boxtimes L) \leq \exists_p \boxtimes K \leq K$$

and, similarly, $\exists_p(\boxtimes K \wedge \boxtimes L) \leq L$. Hence, $\exists_p(\boxtimes K \wedge \boxtimes L) \leq K \wedge L$. Since p was an arbitrary endomorphism, $\boxtimes K \wedge \boxtimes L \leq \boxtimes(K \wedge L)$. $\square$

REMARK 3.8.8. Unlike $\Box$, the operator $\boxtimes$ does not commute with pullbacks along homomorphisms. Let $\Gamma: \mathbf{Set} \twoheadrightarrow \mathbf{Set}$ be the identity functor. We will consider a coequation K over 2 colors, that is, a subset of $UH2 = 2^\omega$, the set of streams over 2. Specifically, let

$$K = \{\underline{0},\, \underline{1}\},$$

where $\underline{0}$ and $\underline{1}$ are the constant streams. Note that K is invariant.

Let $p: H3 \twoheadrightarrow H2$ be the homomorphism induced by the coloring $\overline{p}: 3 \twoheadrightarrow 2$, where

$$\overline{p}(0) = 0, \qquad \overline{p}(1) = 0, \qquad \overline{p}(2) = 1$$

(i.e., $p = H(\overline{p})$). Then p^*K is the set

$$\{\sigma \in 3^\omega \mid \forall n\ \sigma(n) < 2\} \cup \{\underline{2}\}.$$

It is easy to check that

$$\boxtimes p^*K = \{\underline{0},\, \underline{1},\, \underline{2}\} \neq p^*(\boxtimes K) = p^*K.$$

In terms of substitutions, then, it is not the case that, for every homomorphism

$$f\colon\langle B,\,\beta\rangle \longrightarrow \langle A,\,\alpha\rangle\,,$$

$(\boxtimes K)[f(y)/x] = \boxtimes(K[f(y)/x])$.

Next, we show that, for any coequation K over C, $\Box K$ and $\boxtimes K$ define the same covarieties as K. Dually, then, we are proving that, given a set of equations E, the varieties defined by taking the least congruence containing E and by closing E under substitutions are the same as the variety that E defines.

THEOREM 3.8.9. *Let C be given, K a coequation over C.*

$$\mathcal{C}o\mathcal{I}d(K) = \mathcal{C}o\mathcal{I}d(\boxtimes K).$$

PROOF. Since $\boxtimes K \leq K$, clearly $\mathcal{C}o\mathcal{I}d(\boxtimes K) \leq \mathcal{C}o\mathcal{I}d(K)$. For the other inclusion, suppose that $\langle B,\,\beta\rangle \Vdash K$. Let

$$p\colon\langle B,\,\beta\rangle \longrightarrow HC$$

be given. To show that $\mathsf{Im}(p) \leq \boxtimes K$, we will show that, for every endomorphism

$$r\colon HC \longrightarrow HC\,,$$

$\exists_r \mathsf{Im}(p) \leq K$. But, $\exists_r \mathsf{Im}(p) = \mathsf{Im}(r \circ p) \leq K$, since $\langle B,\,\beta\rangle \Vdash K$. $\Box$

THEOREM 3.8.10. *Let C be given and K a coequation over C.*

$$\mathcal{C}o\mathcal{I}d(K) = \mathcal{C}o\mathcal{I}d(\Box K).$$

PROOF. Again, trivially, $\mathcal{C}o\mathcal{I}d(\Box K) \leq \mathcal{C}o\mathcal{I}d(K)$. Let $\langle B,\,\beta\rangle \Vdash K$ and let

$$p\colon\langle B,\,\beta\rangle \longrightarrow HC$$

be given. Then $U\,\mathsf{Im}(p) = \mathsf{Im}(Up) \leq K$ and so, by the adjunction $U \dashv [-]$, $\mathsf{Im}(p) \leq [K]$. Thus,

$$\mathsf{Im}(Up) = U\,\mathsf{Im}(p) \leq U[K] = \Box K.$$

$\Box$

Recall the $[-]$ functor from Section 2.2.1, which takes a regular subobject of $A = U\langle A,\,\alpha\rangle$ to the largest subcoalgebra contained in A. Hence, if K is a coequation over C, then $[K]_{HC}$ is a subcoalgebra of HC. Since it is a coalgebra, in particular, one may ask whether the coalgebra $[K]$ forces the coequation K. In general, this is not the case. However, if K is invariant, then $[K] \Vdash K$, as the following lemma shows.

LEMMA 3.8.11. *Let K be a coequation over injective C. Then $[\boxtimes K] \Vdash K$.*

Proof. Let $p\colon[\boxtimes K]\to HC$ be given. Because HC is regular injective, p extends to a homomorphism $HC\to HC$, as shown in Figure 12. Hence, because

$$\Box \boxtimes K < \boxtimes K$$

and $\boxtimes K$ is invariant, there is a unique map $\Box \boxtimes K \to \boxtimes K$ making the square and thus the lower triangle commute, as desired. $\Box$

$$\begin{tikzcd} UHC \arrow[r, dotted] & UHC \\ \Box \boxtimes K \arrow[u, tail] \arrow[ur, "p"] \arrow[r, dotted] & \boxtimes K \arrow[u, tail] \end{tikzcd}$$

Figure 12. $[\boxtimes K] \Vdash K$.

The final lemma shows the relationship between $\Box$ and $\boxtimes$. One has the idea that $\Box$ and $\boxtimes$ "ought" to commute, but at this point we have not found a general proof of that claim. See, however, Theorem 3.8.14 for a proof that $\boxtimes$ commutes with $\Box$ when Γ preserves non-empty intersections.

Lemma 3.8.12. *For any injective C,*

$$\Box\boxtimes \leq \boxtimes\Box.$$

Proof. By definition of $\boxtimes$, it suffices to show that, for every endomorphism $p\colon HC\to HC$, $\exists_p \Box \boxtimes K \leq \Box K$. We know that, for every p, $\exists_p \Box \boxtimes K \leq \exists_p \boxtimes K \leq K$. Thus, since U commutes with $\exists_p$,

$$U\exists_p[\boxtimes K]_{HC} = \exists_p U[\boxtimes K]_{HC} \leq K,$$

and so $\exists_p[\boxtimes K]_{HC} \leq [K]_{HC}$. Thus,

$$\exists_p \Box \boxtimes K = U\exists_p[\boxtimes K]_{HC} \leq U[K]_{HC} = \Box K.$$

$\Box$

These lemmas allow a simple proof of the invariance theorem.

Theorem 3.8.13 (Invariance theorem). *Let C be injective, $K \leq UHC$. Then $K = \mathcal{C}o\mathcal{I}d(\mathbf{V})$ for some class $\mathbf{V}$ of coalgebras just in case $K = \Box \boxtimes K$.*

Proof. Let $K = \Box \boxtimes K$ and define

$$\mathbf{V} = \{\langle B, \beta\rangle \mid \langle B, \beta\rangle \Vdash K\}.$$

Then, clearly, $\mathbf{V} \Vdash K$. We will show that, if $\mathbf{V} \Vdash L$, then $K \leq L$. From Lemma 3.8.12, we see that

$$K = \Box \boxtimes K = \Box \boxtimes \boxtimes K \leq \boxtimes \Box \boxtimes K = K$$

so $K = \boxtimes K$. From Lemma 3.8.11, we know that $[\boxtimes K] = [K]$ is in $\mathbf{V}$. Consequently, $[K] \Vdash L$ and hence

$$K = \exists_{\mathsf{id}} \Box \boxtimes K \leq L.$$

$\Box$

As we said previously, one suspects that $\Box$ ought to commute with $\boxtimes$. Instead, we have shown (by Lemma 3.8.12) the weaker claim that $\Box\boxtimes \leq \boxtimes\Box$. We have neither a proof that, in general, $\Box\boxtimes = \boxtimes\Box$ nor a counterexample. However, the following theorem gives some progress to the goal. It shows that, if the forgetful functor

$$U_{HC} : \mathsf{SubCoalg}\, HC \longrightarrow \mathsf{RegSub}(UHC)$$

has a left adjoint, as discussed in Section 2.3.

Theorem 3.8.14. *If U_{HC} has a left adjoint, $\langle\rangle_{HC}$, then $\boxtimes\Box = \Box\boxtimes$.*

Proof. To show that $\boxtimes\Box \leq \Box\boxtimes$, it is sufficient (by the adjunction $U\langle\rangle \dashv \Box$) to show that $U\langle\rangle \boxtimes \Box \leq \boxtimes$.

Let $K \leq UHC$. We will show that, for every homomorphism $p : HC \twoheadrightarrow HC$, $\exists_p U\langle\rangle \boxtimes \Box K \leq K$ and conclude (by definition of $\boxtimes$) that $U\langle\rangle \boxtimes \Box K \leq \boxtimes K$. By Theorem 2.2.17 ($\Box$ commutes with pullback along homomorphisms), it suffices to show that

$$\boxtimes \Box K \leq \Box p^* K = p^* \Box K,$$

or, equivalently, $\exists_p \boxtimes \Box K \leq \Box K$. This is immediate from the definition of $\boxtimes$. $\Box$

3.9. Behavioral covarieties and monochromatic coequations

In typical applications of coalgebras in computer science, one is concerned with behavior "up to bisimulation". That is, if two coalgebras behave the same (according to bisimulation equivalence), then we do not distinguish the two, regardless of differences in "internal structure". Thus, one is often concerned with covarieties which are closed under total bisimulations. In this section, we discuss such covarieties, which were first studied in [**GS98**]. For another description of the same class of covarieties, see [**Roş01**]. The material covered here is also found in [**AH00**].

Definition 3.9.1. A *total relation* is a relation for which each projection is epi.

Definition 3.9.2. A *behavioral covariety* is a covariety which is closed under total bisimulations. That is, a covariety $\mathbf{V}$ such that, whenever $\langle A, \alpha\rangle \in \mathbf{V}$ and there is a total bisimulation relating $\langle A, \alpha\rangle$ to $\langle B, \beta\rangle$, then $\langle B, \beta\rangle$ is also in $\mathbf{V}$.

We differ from Gumm on terminology here, as he refers to covarieties closed under total bisimulations as *complete covarieties*.

The following theorem ensures that total bisimulations are the images of total relations in $\mathcal{E}_\Gamma$.

THEOREM 3.9.3. *If $\langle S, \sigma\rangle$ is a relation on $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$. Then $U_{\alpha,\beta}\langle S, \sigma\rangle$ is a total bisimulation iff $\langle S, \sigma\rangle$ is a total relation.*

PROOF. Let $R = U_{\alpha,\beta}\langle S, \sigma\rangle$, with epi projections r_1 and r_2 and let $p : S \twoheadrightarrow R$ be the epi part of the epi-regular mono factorization, $\langle Us_1, Us_2\rangle = \langle r_1, r_2\rangle \circ p$. Then $Us_1 = r_1 \circ p$ and $Us_2 = r_2 \circ p$, so Us_1, Us_2 are epis iff r_1, r_2 are epis, respectively. By Theorem 1.2.13, U preserves and reflects epis. □

Gumm shows that behavioral covarieties over **Set** are definable as coequations over 1. We generalize that result to this setting and show some further equivalences. In particular, the following theorem shows that the behavioral covarieties are exactly the covarieties which are sinks, in the terminology of [**Roş01**].

THEOREM 3.9.4. *Let $\mathbf{V}$ be a covariety of $\mathcal{E}_\Gamma$. The following are equivalent.*

(1) $\mathbf{V}$ is closed under total bisimulations.
(2) $\mathbf{V}$ is closed under domains of epis.
(3) $\mathbf{V}$ is closed under domains of arbitrary homomorphisms.
(4) $\mathbf{V}$ is definable by a coequation over one color (i.e.,

$$\mathbf{V} = \{i : \bullet \rightarrowtriangle H1\}_\perp$$

for some regular mono i).

PROOF. We prove 1⇒2 ⇒3 ⇒1 and 3 ⇔4.

1 ⇒ 2: The graph of epis are total bisimulations.

2 ⇒ 3: Let $f : \langle A, \alpha\rangle \to \langle B, \beta\rangle$ be given, $\langle B, \beta\rangle \in \mathbf{V}$, and take the epi-regular mono factorization, $f = i \circ p$. The domain of i is in $\mathbf{V}$ as a regular subcoalgebra of $\langle B, \beta\rangle$. Hence $\langle A, \alpha\rangle \in \mathbf{V}$.

3 ⇒ 1: Let $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ be given and let $\langle R, \rho\rangle$ be a total bisimulation on $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$. Suppose, further, that $\langle A, \alpha\rangle \in \mathbf{V}$. Then, $\langle R, \rho\rangle \in \mathbf{V}$, since it is the domain of the projection

$$\langle R, \rho\rangle \longrightarrow \langle A, \alpha\rangle .$$

Since $\mathbf{V}$ is closed under codomains of epi homomorphisms, $\langle B, \beta\rangle \in \mathbf{V}$.

3 ⇒ 4: Since $\mathbf{V} \subseteq \{\varepsilon^{\mathbf{V}}_{H1}\}_\perp$, it suffices to show the other inclusion. Let $\langle A, \alpha\rangle$ be given and suppose that $\langle A, \alpha\rangle \perp \varepsilon^{\mathbf{V}}_{H1}$. Then $! : \langle A, \alpha\rangle \to H1$ factors through $\varepsilon^{\mathbf{V}}_{H1}$, and so $\langle A, \alpha\rangle$ is the domain of an arrow into $U^{\mathbf{V}}H^{\mathbf{V}}H1$, which is in $\mathbf{V}$.

$4 \Rightarrow 3$: Let $\mathbf{V} = \{i\}_\perp$, where i is a regular mono into $H1$. Let

$$p: \langle A, \alpha\rangle \longrightarrow \langle B, \beta\rangle$$

be given and suppose $\langle B, \beta\rangle \in \mathbf{V}$. Then $!_\beta: \langle B, \beta\rangle \rightarrowtail H1$ factors through i, say, $!_\beta = i \circ f$. Consequently, $!_\alpha = i \circ f \circ p$. Since $!_\alpha$ is the only map from $\langle A, \alpha\rangle$ to $H1$, it follows that $\langle A, \alpha\rangle \perp i$.

□

REMARK 3.9.5. In the proof of $3 \Rightarrow 1$, we see that if $\langle A, \alpha\rangle$ in $\mathbf{V}$ and $\langle R, \rho\rangle$ is a bisimulation on $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ such that

$$\langle R, \rho\rangle \longrightarrow \langle B, \beta\rangle$$

is epi, then $\langle B, \beta\rangle \in \mathbf{V}$. We do not require that both projections are epis.

EXAMPLE 3.9.6. Very often, the initial Γ-algebra can be realized as a regular subcoalgebra of the final Γ-coalgebra, via the comparison map of Section 1.5.4 (see [**Adá01, Bar93**] for development of this topic). In these cases, the initial algebra can also be viewed as a coequation φ over 1 color.

This provides a useful coequation in the standard examples, allowing one to distinguish between coalgebras consisting of well-founded trees, say, and those which also contain non-well-founded trees.

It is instructive to compare Theorem 3.9.4 to its dual, which says that a variety of algebras is closed under codomains of monos iff it is definable by a set of equations with no variables. See Section 3.9.3 for details.

3.9.1. A covariety closure operation. We can also consider a covariety closure operation, taking a covariety to the least behavioral covariety containing it. Specifically, we define an operator

$$\mathrm{CoVar}(\mathcal{E}_{\mathbb{G}}) \longrightarrow \mathrm{CoVar}(\mathcal{E}_{\mathbb{G}})$$

taking a covariety $\mathbf{V}$ to the collection $\overline{\mathbf{V}}$, where $\langle A, \alpha\rangle \in \overline{\mathbf{V}}$ iff there is some map $f: \langle A, \alpha\rangle \rightarrow \langle B, \beta\rangle$ with $\langle B, \beta\rangle \in \mathbf{V}$, thus closing $\mathbf{V}$ under domains of arbitrary homomorphisms.

It is easy to show that this closure produces another covariety. Hence,

THEOREM 3.9.7. *If $\mathbf{V}$ is a covariety, then $\overline{\mathbf{V}}$ is a behavioral covariety.*

The next theorem states in coequational terms how to obtain $\overline{\mathbf{V}}$. We know that $\mathbf{V}$ is defined by a collection of coequations, in the sense that $\mathbf{V}$ is exactly the class of coalgebras co-orthogonal to a collection of regular monos with cofree codomains. In fact, we can say more about the collection of regular monos — namely, that the regular monos are the components of the counit of a regular mono co-reflection

(Corollary 3.2.8). We show that this counit also gives a defining coequation for $\overline{\mathbf{V}}$. Of course, since $\overline{\mathbf{V}}$ is a behavioral covariety, the only component one needs to consider is that of the final coalgebra.

THEOREM 3.9.8. *Let* $\mathbf{V}$ *be a variety and* $\varepsilon^{\mathbf{V}} : U^{\mathbf{V}} H^{\mathbf{V}} \to 1_{\mathcal{E}_{\mathbb{G}}}$ *be the counit of the associated adjunction*

$$\begin{tikzcd} \mathbf{V} \arrow[r, bend left, "U^{\mathbf{V}}"] \arrow[r, phantom, "\perp"] & \mathcal{E}_{\mathbb{G}} \arrow[l, bend left, "H^{\mathbf{V}}"] \end{tikzcd}$$

Then $\overline{\mathbf{V}} = \{\varepsilon^{\mathbf{V}}_{H1}\}_{\perp}$.

PROOF. Let $\langle A, \alpha\rangle \in \overline{\mathbf{V}}$. Then there is an $f : \langle A, \alpha\rangle \twoheadrightarrow \langle B, \beta\rangle$ such that $\langle B, \beta\rangle \in \mathbf{V}$. Since $\langle B, \beta\rangle \in \mathbf{V}$, clearly $\langle B, \beta\rangle \perp \varepsilon^{\mathbf{V}}_{H1}$. Consequently, $\langle A, \alpha\rangle \perp \varepsilon^{\mathbf{V}}_{H1}$.

On the other hand, if $\langle A, \alpha\rangle \perp \varepsilon^{\mathbf{V}}_{H1}$, then the factorization of $\langle A, \alpha\rangle \twoheadrightarrow H1$ through $\varepsilon^{\mathbf{V}}_{H1}$ is a homomorphism into a coalgebra in $\mathbf{V}$. Hence $\langle A, \alpha\rangle \in \overline{\mathbf{V}}$. □

Note that behavioral covarieties are defined by a single coequation, regardless of any boundedness conditions on Γ.

3.9.2. An example of a non-behavioral covariety. We have given a couple of examples of non-behavioral covarieties previously, including Examples 3.6.11, 3.6.14, 3.6.15 and 3.6.16. We provide an example here that arises from a comparison of categories of streams.

Consider the functors $\mathbb{N} \times -$ and $1 + \mathbb{N} \times -$ on the category **Set**. As usual, we think of coalgebras for these functors as collections of streams over $\mathbb{N}$ (see [**JR97**], for instance). In particular, a coalgebra for $\mathbb{N} \times -$ can be thought of as a collection of infinite streams, closed under the tail destructor. A coalgebra for $1 + \mathbb{N} \times -$ can be understood as a collection of finite or infinite streams over $\mathbb{N}$, again closed under the tail destructor (when defined).

It is clear that the category $\mathbf{Set}_{\mathbb{N}\times -}$ is a full subcategory of $\mathbf{Set}_{1+\mathbb{N}\times -}$. What is less obvious is that one can regard $\mathbf{Set}_{1+\mathbb{N}\times -}$ as a full subcategory of $\mathbf{Set}_{\mathbb{N}\times -}$, and it is this perspective on which we will focus. Define a functor $\mathbf{Set}_{1+\mathbb{N}\times -} \twoheadrightarrow \mathbf{Set}_{\mathbb{N}\times -}$ as follows. If $\langle A, \alpha\rangle$ is a $1 + \mathbb{N} \times -$ coalgebra, then $I(\langle A, \alpha\rangle) = \langle A, \alpha\rangle'$ will be a $\mathbb{N} \times -$ coalgebra. Specifically, let α' be defined by

$$\alpha'(a) = \begin{cases} \langle 0, a\rangle & \text{if } \alpha(a) = * \\ \langle h_\alpha(a) + 1, t_\alpha(a)\rangle & \text{else} \end{cases}$$

(where $h_\alpha(a) = \pi_1 \circ \alpha(a)$ and $t_\alpha = \pi_2 \circ \alpha(a)$ when $\alpha(a) \in \mathbb{N} \times A$). Intuitively, I takes infinite lists to the list one gets by applying successor in each position. For finite lists, I again applies successor in each position and then tacks on 0's at the end. However, the 0's are tacked on in a particular manner — once we hit 0 in the list, the "state" never changes. We stay at the same element of A and continue outputting 0's. This description should lend plausibility to the claim that $\mathbf{V}$ is not behavioral, which we will later prove. The property that a coalgebra stabilizes at a particular state is not a property closed under total bisimulation.

It is routine to check that this defines a functor and, furthermore, that it is full, faithful and regular injective on objects. Let $\mathbf{V}$ be the image of $\mathbf{Set}_{1+\mathbb{N}\times -}$. One could check directly that $\mathbf{V}$ is a covariety, but we prefer to explicitly give a defining coequation (over 2 colors) instead. In keeping with the coloring metaphor, we denote the elements of 2 by red and blue.

Let $\langle h, t\rangle$ be the structure map on $H2$ and define $\varphi \leq UH2$ by

$$\varphi = \{\sigma \in UH2 \mid h(\sigma) = 0 \rightarrow \varepsilon_2(\sigma) = \varepsilon_2 \circ t(\sigma)\}.$$

We will show that $\mathbf{V} = \mathcal{F}rc(\varphi)$.

Suppose that $a \in A$ and $h_\alpha(a) = 0$, but $t_\alpha(a) \neq a$ (i.e., assume $\langle A, \alpha\rangle \notin \mathbf{V}$). Then, we define a coloring p on A by

$$p(b) = \begin{cases} \mathsf{red} & \text{if } a = b \\ \mathsf{blue} & \text{else} \end{cases}$$

Then, let $\widetilde{p}$ be the adjoint transpose of p. We see that

$$h(\widetilde{p}(a)) = h_\alpha(a) = 0,$$

but $\varepsilon_2(\widetilde{p}(a)) = \mathsf{red}$ and

$$\varepsilon(t(\widetilde{p}(a))) = \varepsilon(\widetilde{p}(t(a))) = p(t(a)) = \mathsf{blue}\,.$$

Hence, $\widetilde{p}(a) \notin \varphi$, and so $\langle A, \alpha\rangle \nVdash \varphi$.

On the other hand, suppose that $\langle A, \alpha\rangle \in \mathbf{V}$ and let $p\colon A \twoheadrightarrow 2$ be given. Let $a \in A$ and we will show that $\widetilde{p}(a) \in \varphi$. Accordingly, assume that $h(\widetilde{p}(a)) = 0$. Then, $h_\alpha(a) = 0$ and so $t_\alpha(a) = a$. Consequently, $\varepsilon_2(\widetilde{p}(a)) = \varepsilon_2(t\widetilde{p}(a))$ and so $\widetilde{p}(a) \in \varphi$. Since this holds for any $a \in A$, we see that $\widetilde{p}$ factors through φ and so $\langle A, \alpha\rangle \Vdash \varphi$.

Remark 3.9.9. While this coequation defines the covariety $\mathbf{V}$, it is worth noting that φ is not itself an element of the covariety. Instead, there is a proper regular subcoalgebra of φ which is in the covariety and which also defines $\mathbf{V}$, namely $U \boxtimes [\varphi]$, where $\boxtimes$ is the modal operator from Section 3.8 which takes a coequation to its largest invariant subcoalgebra. This coequation is given by

$$U \boxtimes [\varphi] = \{\sigma \in UH2 \mid \forall n h t^n(\sigma) = 0 \rightarrow t^{n+1}(\sigma) = t^n(\sigma)\}.$$

3.9.3. The dual to behavioral covarieties. In this section, we relate the discussion of behavioral covarieties to categories of algebras. Throughout this section, we assume that $\mathcal{E}$ is a Birkhoff category and Γ preserves regular epis and is a varietor, so that $U\colon \mathcal{E}^\Gamma \twoheadrightarrow \mathcal{E}$ is monadic.

Definition 3.9.10. Let $\mathbf{V}$ be a full subcategory of $\mathcal{E}^{\Gamma}$. We say that $\mathbf{V}$ is an *elementary variety* if $\mathbf{V} = \{p\}^{\perp}$ for some regular epi $p\colon F0 \twoheadrightarrow \bullet$.

Clearly, if $\mathbf{V}$ is an elementary variety, then it is a variety.

In the traditional setting, then, a variety is elementary just in case it is definable by a set of *variable-free* equations. Of course, if the signature has no constants, then this means that the only elementary variety is trivial.

Definition 3.9.11. We say that two Γ-algebras $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ are *constructibly equivalent* just in case $\langle 0\rangle_{\alpha} \cong \langle 0\rangle_{\beta}$ (i.e., just in case the least subalgebra of $\langle A, \alpha\rangle$ is isomorphic to the least subalgebra of $\langle B, \beta\rangle$).

We call this constructible equivalence because it requires that the "constructible" parts of $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ are isomorphic. That is, it requires that those elements of $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ which can be specified by a variable-free term (i.e., by terms in $F0$) satisfy the same equations. This description hints at the relation between constructible equations and elementary varieties. We make the relation explicit in the following theorem.

Theorem 3.9.12. *Let $\mathbf{V}$ be a variety of $\mathcal{E}^{\Gamma}$. The following are equivalent.*

(1) $\mathbf{V}$ is closed under constructible equivalences.
(2) $\mathbf{V}$ is closed under codomains of monos.
(3) $\mathbf{V}$ is closed under codomains of arbitrary homomorphisms.
(4) $\mathbf{V}$ is elementary.

Proof. This theorem is the dual of Theorem 3.9.4. However, we have not dualized closure under total bisimulations directly, since corelations are not familiar objects of study. Instead, we've replaced that condition with the closure under constructible equivalence. We provide the relevant steps.

1 ⇒ 2: Let $\langle A, \alpha\rangle \rightarrowtail \langle B, \beta\rangle$ be given. Then, by Theorem 1.3.7, $\langle 0\rangle_{\alpha}$ is given as the factorization of $!_{\alpha}$, as shown below.

$$\begin{tikzcd}
F0 \arrow[rr, "!_{\alpha}"] \arrow[dr, -{Triangle[open]}] & & \langle A, \alpha\rangle \arrow[r, tail] & \langle B, \beta\rangle \\
& \langle 0\rangle_{\alpha} \arrow[ur, tail] & &
\end{tikzcd}$$

By the uniqueness of regular epi-mono factorizations, $\langle 0\rangle_{\alpha} \cong \langle 0\rangle_{\beta}$, so $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ are constructibly equivalent.

4 ⇒ 1: Let

$$\mathbf{V} = \{p\colon F0 \twoheadrightarrow \langle Q, \nu\rangle\}^{\perp}$$

and let $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ be constructibly equivalent, with $\langle A, \alpha\rangle \in \mathbf{V}$. Then, $!_\alpha$ factors through p, as shown in the diagram below.

$$\begin{tikzcd}
F0 \arrow[r, two heads] \arrow[d, two heads, "p"'] & \langle 0\rangle_\alpha \arrow[r, tail] & \langle A, \alpha\rangle \\
\langle Q, \nu\rangle \arrow[ur, dotted] \arrow[urr, dotted] & &
\end{tikzcd}$$

Because p is regular (and hence, strong), we have the factorization of

$$\langle Q, \nu\rangle \longrightarrow \langle A, \alpha\rangle$$

through $\langle 0\rangle_\alpha \cong \langle 0\rangle_\beta$, as shown. This gives a factorization of $!_\beta$ through $\eta^{\mathbf{V}}_{F0}$, as desired.

$\square$

CHAPTER 4

The internal logic of $\mathcal{E}_{\mathbb{G}}$

It is well-known that, if $\mathcal{E}$ is a topos and a comonad $\mathbb{G}:\mathcal{E}\to\mathcal{E}$ is left exact, the category of coalgebras $\mathcal{E}_{\mathbb{G}}$ is also a topos [**BW85**, Theorem 6.4.1]. Furthermore, Johnstone, et al, strengthened this result in [**JPT⁺98**] by showing that, if $\mathbb{G}$ preserves pullbacks (but not necessarily all finite limits), then $\mathcal{E}_{\mathbb{G}}$ is again a topos. One corollary to these theorems is that there is a natural logic for at least certain categories of coalgebras, the "internal logic" associated with a topos.

In this chapter, we weaken the conditions on $\mathcal{E}$ and $\mathbb{G}$, so that $\mathcal{E}_{\mathbb{G}}$ is not necessarily a topos, while retaining sufficient structure so that we can define an internal logic for $\mathcal{E}_{\mathbb{G}}$. In particular, in this section, we show that if $\mathcal{E}$ is a locally complete logos, with regular epi-regular mono factorizations and coproducts, and $\mathbb{G}$ nearly preserves pullbacks, then $\mathcal{E}_{\mathbb{G}}$ is a locally complete logos. Consequently, we can define a logic $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$ which can be interpreted in $\mathcal{E}_{\mathbb{G}}$. We discuss the internal logic of a an arbitrary locally complete logos in Section 4.1.2.

Since, by assumption, the base category $\mathcal{E}$ is also a locally complete logos, there is an internal logic, $\mathcal{L}(\mathcal{E})$, for it as well. We introduce types for the carriers of coalgebras and a modal operator for the largest subcoalgebra construction in Section 4.2. We also introduce a translation of formulas from $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$ to $\mathcal{L}(\mathcal{E})$ which preserves and reflects valid sequents.

We conclude the chapter by relating the Kripke-Joyal semantics for $\mathcal{L}(\mathcal{E})$ to the definition of coequation forcing given in Chapter 3, and offering a definition of pointwise forcing of coequations which we relate to the comonad associated with a coequation.

Throughout this chapter, we develop the theory for categories of coalgebras for a comonad, rather than coalgebras for an endofunctor. We do this so that the internal logic we develop can be applied to the covarieties from Chapter 3. In fact, the presence of a right adjoint to the forgetful functor plays little role otherwise.

4.1. Preliminary results

We begin with a result found in [**GHS01**]. One could develop the internal logic of $\mathcal{E}_{\mathbb{G}}$ without requiring that the category is extensive, although this would preclude our definitions for coproduct types. We want to exploit coproducts in $\mathcal{E}_{\mathbb{G}}$ by introducing the appropriate types and terms in $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$, and so we begin with a proof that, if $\mathbb{G}$ preserves pullbacks along regular monos, $\mathcal{E}_{\mathbb{G}}$ inherits extensiveness from the base category $\mathcal{E}$. A similar result can be found in [**JPT$^+$98**, Lemma 3.8].

THEOREM 4.1.1. *If $\mathcal{E}$ is extensive, with epi-regular mono factorizations, and $\mathbb{G}$ preserves regular monos and non-empty pullbacks along regular monos, then $\mathcal{E}_{\mathbb{G}}$ is extensive.*

PROOF. Let the commutative diagram in Figure 1 be given. Since co-projections

$$
\begin{tikzcd}
\langle X,\,\chi\rangle \arrow[r] \arrow[d] & \langle Z,\,\zeta\rangle \arrow[d] & \langle Y,\,\upsilon\rangle \arrow[l] \arrow[d] \\
\langle A,\,\alpha\rangle \arrow[r, tail, "\kappa_\alpha"'] & \langle A,\,\alpha\rangle + \langle B,\,\beta\rangle & \langle B,\,\beta\rangle \arrow[l, tail, "\kappa_\beta"]
\end{tikzcd}
$$

FIGURE 1. $\mathcal{E}_{\mathbb{G}}$ is extensive.

in extensive categories are regular monos [**Tay99**, Lemma 5.5.7] and U reflects regular monos, the co-projections κ_α and κ_β are also regular monos.

Hence, U creates pullbacks along co-projections. Since U also creates coproducts, the result follows. □

Until now, we have been satisfied with epi-regular mono factorizations without assuming that these factorizations are stable under pullback. To develop a reasonable internal logic, one wants this stability condition. Without stable factorizations, we would lose basic structural features in the language, including that existentials and joins commute with substitutions.

One way to ensure stable factorizations is to ensure that our epi-regular mono factorizations involve epis which are stable under pullbacks. This is the approach we take here, exploiting results from [**JPT$^+$98**], in which they show that, if $\mathbb{G}$ nearly preserves pullbacks, then $\mathcal{E}_{\mathbb{G}}$ inherits regularity from $\mathcal{E}$.

DEFINITION 4.1.2. A functor $F\colon\mathcal{C}\to\mathcal{D}$ *nearly preserves pullbacks* if, for each pullback $A \times_C B$, $F(A \times_C B)$ covers $FA \times_{FC} FB$, i.e., the canonical isomorphism

$$F(A \times_C B) \longrightarrow FA \times_{FC} FB$$

is a regular epi.

In ibid, they show that, if F nearly preserves pullbacks, then it preserves pullbacks along monos and hence it preserves monos. Using this, they prove that, if the comonad $\mathbb{G}$ nearly preserves pullbacks and $\mathcal{E}$ is regular, then $\mathcal{E}_{\mathbb{G}}$ is regular ([**JPT$^+$98**, Lemma 3.9]). We adapt that result to our setting, in which regular monos are of special interest. Hence, we will assume that the category $\mathcal{E}$ has regular epi-regular mono factorizations. First, we note that this implies that every mono is regular (and every epi is regular, too). While this seems a somewhat strong restriction, it is true in any topos.

LEMMA 4.1.3. *In a category $\mathcal{E}$ with regular epi-regular mono factorizations, every mono (epi,resp.) is regular.*

PROOF. Let $i : A \rightarrowtail B$ be given and take the regular epi-regular mono factorizations $i = j \circ p$. Since p is both regular epi and mono, it is an isomorphism. Dualize to conclude that every epi is regular, too. ☐

THEOREM 4.1.4. *If $\mathcal{E}$ is regular, with regular epi-regular mono factorizations (equivalently, $\mathcal{E}$ regular and every mono regular) and $\mathbb{G}$ nearly preserves pullbacks, then $\mathcal{E}_{\mathbb{G}}$ is regular, with regular epi-regular mono factorizations created by U.*

PROOF. Essentially that from [**JPT$^+$98**]. There, they assume that U preserves monos. Here, we use the fact that $\mathbb{G}$ preserves monos and every mono is regular to conclude that $\mathbb{G}$ preserves regular monos. Thus, we may apply Corollary 1.2.15 to conclude that U preserves regular monos. The rest of the proof goes as in ibid. ☐

COROLLARY 4.1.5. *Under the assumptions of Theorem 4.1.4, every mono in $\mathcal{E}_{\mathbb{G}}$ is regular and hence U preserves and reflects monos.*

We adopt the material that follows from [**Tay99**]. See also [**FS90**].

A logos is a category in which one may interpret first order logic. We sketch how this is done in Section 4.1.2. In the remainder of this section, we show that if $\mathcal{E}$ is a "locally complete" logos (a logos with arbitrary, stable unions), and $\mathbb{G}$ nearly preserves pullbacks, then $\mathcal{E}_{\mathbb{G}}$ is also a locally complete logos.

DEFINITION 4.1.6. A regular category in which finite unions of subobjects exist and are stable under pullbacks and each subobject pullback functor

$$f^* : \mathsf{Sub}(B) \longrightarrow \mathsf{Sub}(A)$$

has a right adjoint is called a *logos*.

DEFINITION 4.1.7. A regular category in which arbitrary unions of subobjects exist and are stable under pullbacks is called a *locally complete logos*.

From [**Tay99**, Definition 5.8.1] and [**Tay99**, Theorem 3.6.9]:

THEOREM 4.1.8. *In a locally complete logos, for each $f : A \twoheadrightarrow B$, the subobject pullback functor*

$$f^* : \mathsf{Sub}(B) \longrightarrow \mathsf{Sub}(A)$$

has a right adjoint (i.e., a locally complete logos is, in particular, a logos).

The following theorem is the main theorem justifying the rest of the chapter. We show that $\mathcal{E}_{\mathbb{G}}$ is a locally complete logos, assuming that $\mathcal{E}$ is (together with some other assumptions). This allows the definition of a first-order internal logic for $\mathcal{E}_{\mathbb{G}}$. Since, by assumption, $\mathcal{E}$ is also a locally complete logos, it, too, has a natural internal logic. We exploit these two logics in Section 4.2.

THEOREM 4.1.9. *Let $\mathcal{E}$ be regular, with regular epi-regular mono factorizations and $\mathbb{G}$ nearly preserve pullbacks (and, hence, $\mathbb{G}$ preserves monos), and suppose further that $\mathcal{E}$ is a locally complete logos with all coproducts. Then $\mathcal{E}_{\mathbb{G}}$ is also a locally complete logos.*

PROOF. The forgetful functor creates unions and pullbacks along monos. Thus, if $\{\langle A_i, \alpha_i\rangle\}_{i\in I}$ is a family of subcoalgebras of $\langle B, \beta\rangle$ and $f:\langle C, \gamma\rangle \to \langle B, \beta\rangle$ is a $\mathbb{G}$-homomorphism, then

$$Uf^* \bigcup \langle A_i, \alpha_i\rangle = f^* \bigcup A_i = \bigcup f^* A_i = U \bigcup f^* \langle A_i, \alpha_i\rangle,$$

and so $f^* \bigcup \langle A_i, \alpha_i\rangle = \bigcup f^* \langle A_i, \alpha_i\rangle$. □

We can give an explicit definition of the functor $\forall_f$ in terms of $[-]$, U and the functor $\forall_{Uf}$ in $\mathcal{E}$. Since we need this characterization in Theorem 4.2.5, we include it here.

THEOREM 4.1.10. *For any homomorphism $f:\langle A, \alpha\rangle \to \langle B, \beta\rangle$,*

$$\forall_f = [-]_\beta \circ \forall_{Uf} \circ U_\alpha.$$

PROOF. Because $f^*\forall_f \le 1$ and $[-]$ commutes with pullbacks of homomorphisms (Corollary 2.2.8), we have, for every $\langle C, \gamma\rangle \le \langle A, \alpha\rangle$,

$$f^*[\forall_f C]_\beta = [f^*\forall_f C]_\alpha \le [C]_\alpha = \langle C, \gamma\rangle.$$

Hence, $[\forall_f C]_\beta \le \forall_f \langle C, \gamma\rangle$.

Conversely,

$$f^* U_\beta \forall_f \langle C, \gamma\rangle = U_\alpha f^* \forall_f \langle C, \gamma\rangle \le U_\alpha \langle C, \gamma\rangle = C,$$

and so $\forall_f \langle C, \gamma\rangle \le [\forall_f C]_\beta$. □

We summarize the results of Theorems 2.2.5, 2.2.5 and 2.2.6 in the following corollary.

COROLLARY 4.1.11. *The forgetful functor $U_\alpha : \mathsf{RegSub}(\langle A, \alpha\rangle) \to \mathsf{RegSub}(A)$ preserves $\wedge$, $\vee$, $\exists$, $\perp$ and $\top$ (but not $\forall$, $\to$ or $\neg$). That is, for any subcoalgebras $\langle P, \rho\rangle, \langle Q, \nu\rangle \le \langle A, \alpha\rangle$, we have*

(1) $U_\alpha(\langle P, \rho\rangle \wedge \langle Q, \nu\rangle) = P \wedge Q$

(2) $U_\alpha(\langle P, \rho\rangle \vee \langle Q, \nu\rangle) = P \vee Q$
(3) For every homomorphism $f : \langle A, \alpha\rangle \to \langle B, \beta\rangle$, $\exists_f \langle P, \rho\rangle = \exists_{Uf} P$.
(4) $U_\alpha\langle A, \alpha\rangle = A$ *and* $U_\alpha\langle 0, !\rangle = 0$.

In other words, U_α "almost" preserves *geometric logic* (see [**LM92**, Chapter X]). The situation is complicated by the fact that U_α does not, in general, preserve finite limits. Thus, it doesn't preserve the interpretation of contexts $\Gamma = x_1 : T_1, \ldots, x_n : T_n$, which complicates the translation of formulas in the internal logic of $\mathcal{E}_{\mathbb{G}}$ into formulas in the internal logic of $\mathcal{E}$. Also, it doesn't preserve equalizers, so equations in $\mathcal{E}_{\mathbb{G}}$ are not translated to equations in $\mathcal{E}$. We will see how to avoid these difficulties in Section 4.2, where we define a translation of formulas from $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$ to related formulas in $\mathcal{L}(\mathcal{E})$.

4.1.1. A weak regular subobject classifier. In this section, we show that if $\mathcal{E}$ has a weak regular subobject classifier, then so does $\mathcal{E}_{\mathbb{G}}$. This section is self-contained, in the sense that we do not exploit the weak regular subobject classifier when we develop the internal logic. Throughout, we assume that $\mathcal{E}$ is almost co-regular and $\mathbb{G}$ preserves regular monos.

Definition 4.1.12. Let $\Omega \in \mathcal{E}$ and $\mathsf{true} : 1 \to \Omega$ be given. We say that Ω (or the pair $\langle \Omega, \mathsf{true}\rangle$) is a *weak regular subobject classifier* if, for every regular mono $P \rightarrowtail A$, there is a (not necessarily unique) $A \to \Omega$ such that the diagram below is a pullback.

$$\begin{tikzcd} P \arrow[r] \arrow[d, tail] \arrow[dr, phantom, very near start, "\lrcorner"] & 1 \arrow[d, "\mathsf{true}"] \\ A \arrow[r] & \Omega \end{tikzcd}$$

Theorem 4.1.13. *Let Ω in $\mathcal{E}$ be a weak regular subobject classifier. Then $H\Omega$ is a weak regular subobject classifier in $\mathcal{E}_{\mathbb{G}}$.*

Proof. Let $\langle P, \rho\rangle \leq \langle A, \alpha\rangle$. We will show that there is a homomorphism $\langle A, \alpha\rangle \to H\Omega$ such that the front face of Figure 2 is a pullback.

Let $r : A \to \Omega$ be a classifying map for p in $\mathcal{E}$ and let $\widetilde{r} : \langle A, \alpha\rangle \to H\Omega$ be the adjoint transpose of r, as in Figure 2. A quick diagram chase confirms that the front face of the prism commutes.

Suppose that $g : \langle B, \beta\rangle \to \langle A, \alpha\rangle$ satisfies $\widetilde{r} \circ g = H\mathsf{true} \circ !$. Then

$$r \circ Ug = \varepsilon_\Omega \circ U(\widetilde{r} \circ g) = \varepsilon_\Omega \circ UH\mathsf{true} \circ U! = \mathsf{true} \circ !,$$

and so $\mathsf{Im}(g) \leq P$. Corollary 1.2.10 ensures that the factorization of g through P is a homomorphism. □

Corollary 4.1.14. *Suppose $\mathcal{E}$ is regular and every mono of $\mathcal{E}$ is regular. Further suppose that $\mathbb{G}$ nearly preserves pullbacks. Then $\mathcal{E}_{\mathbb{G}}$ has a weak subobject classifier.*

$$
\begin{tikzcd}
 & 1 \arrow[dd] & \\
P \arrow[ur] \arrow[rr] \arrow[dd, tail] & & UH1 \arrow[ul, "\varepsilon_1"'] \arrow[dd, "UH\mathsf{true}"] \\
 & \Omega & \\
A \arrow[ur, "r"] \arrow[rr, "U\widetilde{r}"'] & & UH\Omega \arrow[ul, "\varepsilon_\Omega"']
\end{tikzcd}
$$

FIGURE 2. $H\Omega$ is a weak regular subobject classifier

PROOF. Apply Corollary 4.1.5. $\square$

The presence of a regular subobject classifier (not weak) in $\mathcal{E}$ is not sufficient to ensure that $\mathcal{E}_{\mathbb{G}}$ has a regular subobject classifier in general. To see this, consider a homomorphism $\widetilde{r}:\langle A,\, \alpha\rangle \to H\Omega$, and let $r: A \to \Omega$ with P the subobject of A characterized by r, as in Figure 2. Then P is the pullback of $UH\mathsf{true}$ along $U\widetilde{r}$. It is easy to check that $\Box_\alpha P$ is the pullback of $H\mathsf{true}$ along $\widetilde{r}$. Thus, for any homomorphisms

$$
p:\langle A,\, \alpha\rangle \longrightarrow H\Omega,
$$
$$
q:\langle A,\, \alpha\rangle \longrightarrow H\Omega,
$$

we see that p and q classify the same subcoalgebra just in case $\Box P = \Box Q$, where $\varepsilon_\Omega \circ p$ classifies P and $\varepsilon_\Omega \circ q$ classifies Q.

This observation does give a canonical choice for a characteristic map for a subcoalgebra. Given $\langle P,\, \rho\rangle \leq \langle A,\, \alpha\rangle$, as in Theorem 4.1.13, let $\widetilde{r}$ be the transpose of the (unique) characteristic map of P in $\mathcal{E}$. Then, $\widetilde{r}$ is minimal in the sense that, if s is any other characteristic map for $\langle P,\, \rho\rangle$, then (the object classified by) $\varepsilon_\Omega \circ U\widetilde{r}$ is smaller than (the object classified by) $\varepsilon_\Omega \circ Us$.

4.1.2. The internal logic of a logos. Given a locally complete logos $\mathcal{C}$, one can define a first order language $\mathcal{L}(\mathcal{C})$ which can be interpreted in $\mathcal{C}$. The first order intuitionistic logic is sound under this interpretation. Applying this result to the current setting, this leads to *two* first order languages. On the one hand, the base category $\mathcal{E}$ is, by assumption, a locally complete logos and thus we may define a language $\mathcal{L}(\mathcal{E})$ and an interpretation of the language in the category $\mathcal{E}$. On the other hand, $\mathcal{E}_{\mathbb{G}}$ is also a locally complete logos and so we may define a language $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$ over $\mathcal{E}_{\mathbb{G}}$. In Section 4.2, we will translate formulas in the language $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$ to formulas in $\mathcal{L}(\mathcal{E})$

In this section, we will show how one defines a first order language $\mathcal{L}(\mathcal{C})$ for any distributive, locally complete logos $\mathcal{C}$ with coproducts (for the coproduct types below). This construction applies to the categories $\mathcal{E}$ and $\mathcal{E}_{\mathbb{G}}$, yielding the languages $\mathcal{L}(\mathcal{E})$ and $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$.

See any of [**But98, Bor94, LM92, Tay99, LS86**] for presentations of the internal logic of a category.

The language $\mathcal{L}(\mathcal{C})$ is a typed first order language. We write $x\colon T$ to indicate that x is a variable of type T (we assume a countable set of variables). A *context* Γ is a finite list of such declarations. We write $\Gamma, x\colon T$ to indicate the context Γ with a new declaration for x. Whenever we write this, we assume that x does not already occur in Γ. We write $\Gamma \,|\, t\colon T$ to indicate that, in context Γ, the term t is of type T. This notation presumes that the free variables of t appear in Γ. We write $\Gamma \,|\, \varphi$ to indicate that φ is a well-formed formula in context Γ.

For each object $C \in \mathcal{C}$, we define a type C in $\mathcal{L}(\mathcal{C})$. For each pair of types S and T, we define types $S \times T$ and $S + T$. The types are interpreted as objects in $\mathcal{C}$ in the obvious way. I.e., $[\![C]\!] = C$, $[\![S \times T]\!] = [\![S]\!] \times [\![T]\!]$, etc. The type formation rules and interpretation of types are summarized in Table 1.

Type formation rule	Interpretation
C	C
$S \times T$	$[\![S]\!] \times [\![T]\!]$
$S + T$	$[\![S]\!] + [\![T]\!]$
1	1

TABLE 1. The inductive definition of types.

A context inherits its interpretation from the terms, so that

$$[\![x_1\colon T_1, x_2\colon T_2, \ldots, x_n\colon T_n]\!] = [\![T_1]\!] \times [\![T_2]\!] \times \ldots \times [\![T_n]\!].$$

The empty context is, of course, interpreted as 1, the final object of $\mathcal{C}$. We want to treat the contexts as unordered, so that we don't differentiate between the contexts

$$\Gamma = x\colon S, y\colon T \text{ and } \Delta = y\colon T, x\colon S.$$

We may do this by assuming an ordering on the types, so that there is a canonical representative for each equivalence class of contexts (and so that a context is interpreted as its representative is). None of this is crucial in what follows, but it simplifies the presentation.

A term $t\colon T$ in context Γ is interpreted as a function

$$[\![\Gamma \,|\, t\colon T]\!] = [\![t]\!]\colon [\![\Gamma]\!] \longrightarrow [\![T]\!].$$

We omit the types and write $[\![\Gamma \mid t]\!]$, or just $[\![t]\!]$, when convenient. For each type T, and each variable x, we have the term formation rule

$$\Gamma, x\colon T \mid x\colon T.$$

We interpret variables as the projection

$$[\![\Gamma]\!] \times [\![T]\!] \xrightarrow{\pi_T} [\![T]\!].$$

For each arrow $f\colon [\![S]\!] \to [\![T]\!]$, we have also a term formation rule

$$\Gamma, x\colon S \mid fx\colon T,$$

and an interpretation

$$[\![\Gamma, x \mid fx]\!] = f \circ [\![\Gamma, x \mid x]\!] \colon [\![\Gamma]\!] \times [\![S]\!] \longrightarrow [\![T]\!].$$

In addition to variables and terms for each function symbol, we include the following term formation rules in Table 2. In what follows, we let $t[s/x]$ denote the result of substituting the term s for the variable x in term t, where this operation is defined inductively as usual. Similarly, $\varphi[s/x]$ denotes the substitution of s for x in the formula φ, where this is defined as usual.

Term formation rule	Interpretation
$\Gamma, x\colon T \mid x\colon T$	$\pi_T\colon [\![\Gamma]\!] \times [\![T]\!] \to [\![T]\!]$
$\Gamma, x\colon S \mid fx\colon T$	$f \circ [\![x]\!]$
$\Gamma \mid *\colon 1$	$!_{[\![\Gamma]\!]}\colon [\![\Gamma]\!] \to [\![1]\!]$
$\Gamma, x\colon S, y\colon T \mid (x,\, y)\colon S \times T$	$\langle [\![x]\!],\, [\![y]\!] \rangle$
$\Gamma, x\colon S \times T \mid \pi_1 x\colon S$	$\pi_1 \circ [\![x]\!]$
$\Gamma, x\colon S \times T \mid \pi_2 x\colon T$	$\pi_2 \circ [\![x]\!]$
$\Gamma, x\colon S \mid \mathsf{inl}\, x\colon S + T$	$\kappa_1 \circ [\![x]\!]$
$\Gamma, y\colon T \mid \mathsf{inr}\, y\colon S + T$	$\kappa_1 \circ [\![x]\!]$
$\dfrac{\Gamma, x\colon S \mid s\colon U \qquad \Gamma, y\colon T \mid t\colon U}{\Gamma, z\colon S + T \mid \mathsf{case}\ z\ \mathsf{of}\ x \Rightarrow s,\ y \Rightarrow t\colon U}$	$[[\![s]\!],\, [\![t]\!]]$
$\dfrac{\Gamma, x\colon S \mid t\colon T \qquad \Gamma \mid s\colon S}{\Gamma \mid t[s/x]\colon T}$	$[\![t]\!] \circ \langle \mathsf{id}_{[\![\Gamma]\!]},\, [\![s]\!] \rangle$

TABLE 2. Term formation rules for $\mathcal{L}(\mathcal{C})$.

REMARK 4.1.15. In the interpretation of the case statement in Table 2, we implicitly use the isomorphism

$$[\![\Gamma]\!] \times [\![S + T]\!] \cong ([\![\Gamma]\!] \times [\![S]\!]) + ([\![\Gamma]\!] \times [\![T]\!]).$$

REMARK 4.1.16. It is easy to verify that, if $\Gamma \mid t\colon T$, then $\Gamma, \Delta \mid t\colon T$. Furthermore,

$$[\![\Gamma, \Delta \mid t]\!] = [\![\Gamma \mid t]\!] \circ \pi_{[\![\Gamma]\!]}.$$

A formula φ in context Γ is interpreted as a subobject of $[\![\Gamma]\!]$. We give the inductive definition of the class of formulas of $\mathcal{L}(\mathcal{C})$ together with their interpretations in Table 3.

Formula formation rule	Interpretation
$x\colon T \mid \varphi_P(x)$	$P \rightarrowtail [\![T]\!]$
$x\colon T, y\colon T \mid x = y$	Δ_T
$\Gamma \mid \top$	$[\![T]\!] \xrightarrow{\mathsf{id}} [\![T]\!]$
$\Gamma \mid \bot$	$0 \rightarrowtail [\![T]\!]$
$\dfrac{\Gamma \mid \varphi \qquad \Gamma \mid \psi}{\Gamma \mid \varphi \wedge \psi}$	$[\![\varphi]\!] \wedge [\![\psi]\!]$
$\dfrac{\Gamma \mid \varphi \qquad \Gamma \mid \psi}{\Gamma \mid \varphi \to \psi}$	$[\![\varphi]\!] \to [\![\psi]\!]$
$\dfrac{\Gamma, x\colon T \mid \varphi}{\Gamma \mid \exists_{x\colon T}\varphi}$	$\exists_{\pi_{[\![\Gamma]\!]}}[\![\varphi]\!]$
$\dfrac{\Gamma, x\colon T \mid \varphi}{\Gamma \mid \forall_{x\colon T}\varphi}$	$\forall_{\pi_{[\![\Gamma]\!]}}[\![\varphi]\!]$
$\dfrac{\Gamma, x\colon T \mid \varphi \qquad \Gamma \mid t\colon T}{\Gamma \mid \varphi[t/x]}$	$\langle \mathsf{id}_{[\![\Gamma]\!]}, [\![t]\!]\rangle^*[\![\varphi]\!]$
$\dfrac{\Gamma \mid \varphi}{\Gamma, \Delta \mid \varphi}$	$(\pi_{[\![\Gamma]\!]})^*[\![\varphi]\!]$

TABLE 3. Formula formation and interpretation

The following theorem is standard. We omit the proof.

THEOREM 4.1.17. *For any $x\colon S \mid t\colon T$ and any formula $x\colon S \mid \varphi$,*

$$[\![y\colon T \mid \exists_{x\colon S}(t(x) = y \wedge \varphi(x))]\!] = \exists_{[\![t]\!]}[\![\varphi]\!],$$
$$[\![y\colon T \mid \forall_{x\colon S}(t(x) = y \to \varphi(x))]\!] = \forall_{[\![t]\!]}[\![\varphi]\!].$$

We use a Gentzen-style proof system, although we allow only a single formula as the antecedent of the sequent. A sequent comes in context, where the context applies to both the antecedent and consequent. Thus, a sequent

$$\Gamma \mid \varphi \vdash \psi$$

is understood as the assertion that $\Gamma \mid \varphi$ entails $\Gamma \mid \psi$.

Accordingly, a sequent $\Gamma \mid \varphi \vdash \psi$ is *valid* (written $\models \Gamma \mid \varphi \vdash \psi$) just in case $[\![\Gamma \mid \varphi]\!] \leq [\![\Gamma \mid \psi]\!]$ (as subobjects of $[\![\Gamma]\!]$).

The following are sound rules of inference for $\mathcal{L}(\mathcal{E})$. We just write the sequent for axioms, and we write

$$\frac{\Gamma \mid \varphi \vdash \psi}{\Delta \mid \vartheta \vdash \chi}$$

to indicate a rule that, from $\varphi \vdash \psi$, one can infer $\vartheta \vdash \chi$. We denote equivalences with a double underline, so that

$$\frac{\Gamma \mid \varphi \vdash \psi}{\overline{\Delta \mid \vartheta \vdash \chi}}$$

means that from $\varphi \vdash \psi$, one can infer $\vartheta \vdash \chi$ and also from $\vartheta \vdash \chi$, one can infer $\varphi \vdash \psi$. Structural rules:

(Str1) $\Gamma \mid \varphi \vdash \varphi$
(Str2) $\dfrac{\Gamma \mid \varphi \vdash \psi \qquad \Gamma \mid \psi \vdash \vartheta}{\Gamma \mid \varphi \vdash \vartheta}$
(Str3) $\dfrac{\Gamma, x\colon T \mid \varphi \vdash \psi \qquad \Gamma \mid t\colon T}{\Gamma \mid \varphi[t/x] \vdash \psi[t/x]}$

Logical rules:

(Log1) $\Gamma \mid \varphi \vdash \top$
(Log2) $\dfrac{\Gamma \mid \varphi \vdash \psi \qquad \Gamma \mid \varphi \vdash \vartheta}{\overline{\Gamma \mid \varphi \vdash \psi \wedge \vartheta}}$
(Log3) $\dfrac{\Gamma \mid \varphi \vdash \psi \qquad \Gamma \mid \vartheta \vdash \psi}{\overline{\Gamma \mid \varphi \vee \vartheta \vdash \psi}}$
(Log4) $\dfrac{\Gamma \mid \varphi \vdash \psi \to \vartheta}{\overline{\Gamma \mid \varphi \wedge \psi \vdash \vartheta}}$
(Log5) $\dfrac{\Gamma \mid \varphi \vdash \forall_{x\colon T}\psi}{\overline{\Gamma, x\colon T \mid \varphi \vdash \psi}}$
(Log6) $\dfrac{\Gamma \mid \exists_{x\colon T}\varphi \vdash \psi}{\overline{\Gamma, x\colon T \mid \varphi \vdash \psi}}$

Equality:

(Eq1) $\Gamma \mid \top \vdash x = x$
(Eq2) $\Gamma \mid x_1 = x_2 \vdash x_2 = x_1$
(Eq3) $\Gamma \mid x_1 = x_2 \wedge x_2 = x_3 \vdash x_1 = x_3$
(Eq4) $\Gamma \mid x_1 = x_2 \vdash t(x_1) = t(x_2)$
(Eq5) For each atomic formula φ_P, $\Gamma \mid x_1 = x_2 \wedge \varphi_P(x_1) \vdash \varphi_P(x_2)$

Pairing:

(Pr1) $\Gamma, x\colon 1 \mid \top \vdash x = *$
(Pr2) $\Gamma \mid x_1 = y_1 \wedge x_2 = y_2 \vdash (x_1, x_2) = (y_1, y_2)$
(Pr3) $\Gamma, z\colon S \times T \mid \top \vdash z = (\pi_1 z, \pi_2 z)$
(Pr4) $\Gamma, x\colon S y\colon T \mid \top \vdash \pi_1(x, y) = x \wedge \pi_2(x, y) = y$

As usual, one introduces tupling and projections π_i^n for arbitrary finite products and shows that rules (Pr3) and (Pr4) generalize to theorems

$$\Gamma, z\colon T_1 \times \ldots \times T_n \mid \top \vdash z = \langle \pi_1^n z, \ldots, \pi_n^n z \rangle$$
$$\Gamma, x_1\colon T_1, \ldots, x_n\colon T_n \mid \top \vdash \pi_i^n \langle x_1, \ldots, x_n \rangle = x_i$$

Co-pairing:

(CoPr1) $\Gamma, z\colon S \mid \top \vdash (\mathsf{case}\ \mathsf{inl}\, z\ \mathsf{of}\ x \Rightarrow s,\ y \Rightarrow t) = s[z/x]$
(CoPr2) $\Gamma, z\colon T \mid \top \vdash (\mathsf{case}\ \mathsf{inr}\, z\ \mathsf{of}\ x \Rightarrow s,\ y \Rightarrow t) = t[z/y]$
(CoPr3) $\Gamma, z\colon S + T \mid \top \vdash (\mathsf{case}\ z\ \mathsf{of}\ x \Rightarrow \mathsf{inl}\, x,\ y \Rightarrow \mathsf{inr}\, y) = z$

4.1.3. An example using the internal logic. In this section, we use the internal logic to offer an alternate approach to some of the results in Section 3.9. In Theorem 3.9.4, we showed that, $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ are related by a coalgebraic relation $\langle R, \rho \rangle$ such that the projection $r_\alpha : \langle R, \rho \rangle \twoheadrightarrow \langle A, \alpha \rangle$ is epi just in case $\langle A, \alpha \rangle$ forces any coequations over 1 that $\langle B, \beta \rangle$ forces. Now, there is a relation $\langle R, \rho \rangle$ whose projection to $\langle A, \alpha \rangle$ is epi if and only if the projection

$$\pi_1 : \langle A, \alpha \rangle \times \langle B, \beta \rangle \longrightarrow \langle A, \alpha \rangle$$

is epi. Also,

$$\{\varphi \leq H1 \mid \langle B, \beta \rangle \Vdash \varphi\} \subseteq \{\varphi \leq H1 \mid \langle A, \alpha \rangle \Vdash \varphi\}$$

just in case $\mathsf{Im}(!_\alpha) \leq \mathsf{Im}(!_\beta)$. Thus, we could restate this part of Theorem 3.9.4 as follows.

THEOREM 4.1.18. *The projection* $\pi_1 : \langle A, \alpha \rangle \times \langle B, \beta \rangle \twoheadrightarrow \langle A, \alpha \rangle$ *is epi just in case* $\mathsf{Im}(!_\alpha) \leq \mathsf{Im}(!_\beta)$.

It is easy to show (see [**LS86**]) that, in any locally complete logos, a map $f : A \twoheadrightarrow B$ is epi just in case

$$\models y\colon B \mid \top \vdash \exists_{x\colon A} fx = y.$$

Similarly, it is immediate from the definition of the semantics that $\mathsf{Im}(!_A) \leq \mathsf{Im}(!_B)$ just in case

$$\models \exists_{x\colon A} \top \vdash \exists_{y\colon B} \top.$$

So, we can regard this fact more generally as a fact about locally complete logoses[1] (rather than a fact about categories of coalgebras). Stated in the internal logic, Theorem 4.1.18 can be expressed as follows.

THEOREM 4.1.19. $x\colon S \mid \top \vdash \exists_{z\colon S \times T}(\pi_1 z = x)$ *just in case* $\exists_{x\colon S} \top \vdash \exists_{y\colon T} \top$.

[1] We can actually weaken the requirements on the category, since the proofs don't involve universal quantification. A prelogos (see [**Tay99**]) should suffice.

Proof. From $x\colon S, z\colon S\times T \mid \exists_{y\colon T}\top \vdash \exists_{y\colon T}\top$, we infer

$$x\colon S, y\colon T, z\colon S\times T \mid \top \vdash \exists_{y\colon T}\top.$$

Since any formula proves $\top$, by the cut rule (i.e., (Str2)) we have

$$x\colon S, y\colon T, z\colon S\times T \mid (\pi_1 z = x) \vdash \exists_{y\colon T}\top.$$

Substituting $\pi_2 z$ for y (which does not appear free in the sequent), we infer

$$x\colon S, z\colon S\times T \mid (\pi_1 z = x) \vdash \exists_{y\colon T}\top,$$

and hence $x\colon S \mid \exists_{z\colon S\times T}(\pi_1 z = x) \vdash \exists_{y\colon T}\top$. Now, assuming

$$x\colon S \mid \top \vdash \exists_{z\colon S\times T}(\pi_1 z = x),$$

we see $x\colon S \mid \top \vdash \exists_{y\colon T}\top$ and hence $\exists_{x\colon S}\top \vdash \exists_{y\colon T}\top$.

For the other direction, we use the axiom

$$x\colon S, y\colon T, z\colon S\times T \mid z = \langle x,\, y\rangle \wedge \pi_1\langle x,\, y\rangle = x \vdash (\pi_1 z = x) \qquad \text{(Eq5)}$$

and the theorem $x\colon S, y\colon T, z\colon S\times T \mid (\pi_1 z = x) \vdash \exists_{z\colon S\times T}(\pi_1 z = x)$ to infer

$$x\colon S, y\colon T, z\colon S\times T \mid z = \langle x,\, y\rangle \wedge \pi_1\langle x,\, y\rangle = x \vdash \exists_{z\colon S\times T}(\pi_1 z = x)$$

and thus

$$x\colon S, y\colon T \mid \langle x,\, y\rangle = \langle x,\, y\rangle \wedge \pi_1\langle x,\, y\rangle = x \vdash \exists_{z\colon S\times T}(\pi_1 z = x).$$

Since $\top$ proves the antecedent, an application of cut yields

$$x\colon S, y\colon T \mid \top \vdash \exists_{z\colon S\times T}(\pi_1 z = x)$$

and hence $x\colon S \mid \exists_{y\colon T}\top \vdash \exists_{z\colon S\times T}(\pi_1 z = x)$. Under the assumption that $\exists_{x\colon S}\top \vdash \exists_{y\colon T}\top$, we see that $x\colon S \mid \top \vdash \exists_{y\colon T}\top$ and so another application of cut completes the proof. $\square$

4.2. Transfer principles

Throughout this section, we assume that $\mathcal{E}$ is an extensive, well-powered, locally complete logos with regular epi-regular mono factorizations and all coproducts. We also assume that $\mathbb{G}$ nearly preserves pullbacks, so that the category $\mathcal{E}_{\mathbb{G}}$ is also an extensive, locally complete logos with all coproducts.

Given a locally complete logos $\mathcal{E}$, Section 4.1.2 constructs a first order logic that can be naturally interpreted in $\mathcal{E}$. More generally, given a first order language $\mathcal{L}$, an interpretation of $\mathcal{L}$ in a locally complete logos $\mathcal{E}$ consists of an assignment $[\![-]\!]$ which

- assigns to each type T an object $[\![T]\!]$ of $\mathcal{E}$;
- assigns to each term $\Gamma \mid t\colon T$ an arrow $[\![t]\!]\colon [\![\Gamma]\!] \rightarrowtail [\![T]\!]$;
- assigns to each formula $\Gamma \mid \varphi$ a regular subobject $[\![\varphi]\!]$ of $[\![\Gamma]\!]$ $(= [\![T_1]\!] \times \ldots \times [\![T_n]\!])$.

4.2.1. Translation of types. We augment the language $\mathcal{L}(\mathcal{E})$ by adding types $\ulcorner T \urcorner$ for each type T in $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$. The interpretation of $\ulcorner T \urcorner$ is given by

$$[\![\ulcorner T \urcorner]\!] = U[\![T]\!].$$

Thus, we have the following interpretations. Notice that by introducing new types for

Translation type	Interpretation
$\ulcorner \alpha : A \to GA \urcorner$	A
$\ulcorner S \times T \urcorner$	$U([\![S]\!] \times [\![T]\!])$
$\ulcorner S + T \urcorner$	$U[\![S]\!] + U[\![T]\!]$
$\ulcorner 1 \urcorner$	$UH1$

TABLE 4. The interpretation of translated types.

each type T in $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$, we can distinguish between coalgebras with the same carrier. That is, if

$$\alpha : A \longrightarrow \mathbb{G}A,$$
$$\alpha' : A \longrightarrow \mathbb{G}A$$

are distinct structure maps for A, then we have two types $\ulcorner \alpha \urcorner$ and $\ulcorner \alpha' \urcorner$ in $\mathcal{L}(\mathcal{E})$, both of which are interpreted as A.

The translation of a context $\Gamma = x_1 \colon T_1, \ldots, x_n \colon T_n$ is given by

$$\ulcorner \Gamma \urcorner = z \colon \ulcorner T_1 \times \ldots \times T_n \urcorner.$$

We add the variable z of type $\ulcorner T_1 \times \ldots \times T_n \urcorner$ because the forgetful functor U does not, in general, preserve products. This translation is motivated by the observation that, given $x_1 \colon T_1, \ldots, x_n \colon T_n \mid t \colon T$, then we have

$$z \colon T_1 \times \ldots \times T_n \mid t[\pi_1 z/x_1] \ldots [\pi_n z/x_n] \colon T,$$

and that this term is provably equivalent to the original t (in the sense that, if we substitute $\langle x_1, \ldots, x_n \rangle$ for z, then the result is equal to the original term t).

For readability, we abuse notation and denote the translated product

$$\ulcorner T_1 \times \ldots \times T_n \urcorner \; (= U(\ulcorner T_1 \urcorner \times \ldots \times \ulcorner T_n \urcorner))$$

by $\ulcorner \Gamma \urcorner$, where the meaning of $\ulcorner \Gamma \urcorner$ should be clear from context. Thus, we write

$$\ulcorner \Gamma \urcorner = z \colon \ulcorner \Gamma \urcorner,$$

where the translation on the left is the translation of the context and the translation on the right is the translation of the associated product.

4.2.2. Translation of terms. For each homomorphism f in $\mathcal{E}_{\mathbb{G}}$, we define $\ulcorner f \urcorner = Uf$. We refer to Table 5 for the translation of terms. We write $\ulcorner t \urcorner$ for the translation of a term t. So, for instance,

$$\ulcorner \Gamma, x\colon T \mid fx \urcorner = z\colon \ulcorner \Gamma, T \urcorner \mid \ulcorner f \circ \pi_S \urcorner z,$$

where the $\ulcorner - \urcorner$ on the right hand side refers to the function symbol for $U(f \circ \pi_i)$ in $\mathcal{L}(\mathcal{E})$. For each term-in-context $\Gamma \mid t\colon T$ of $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$, there is a corresponding translation $\ulcorner \Gamma \urcorner \mid \ulcorner t \urcorner\colon \ulcorner T \urcorner$.

Term formation rule	Translation
$\Gamma, x\colon T \mid x\colon T$	$z\colon \ulcorner \Gamma, T \urcorner \mid \ulcorner \pi_T \urcorner z\colon \ulcorner T \urcorner$
$\Gamma, x\colon S \mid fx\colon T$	$z\colon \ulcorner \Gamma, S \urcorner \mid \ulcorner f \circ \pi_S \urcorner z\colon \ulcorner T \urcorner$
$\Gamma \mid *\colon 1$	$z\colon \ulcorner \Gamma \urcorner \mid \ulcorner ! \urcorner\colon \ulcorner 1 \urcorner$
$\dfrac{\Gamma \mid s\colon S \qquad \Gamma \mid t\colon T}{\Gamma \mid \langle s,\, t\rangle\colon S \times T}$	$z\colon \ulcorner \Gamma \urcorner \mid \ulcorner [\![\langle s,\, t\rangle]\!] \urcorner z\colon \ulcorner S \times T \urcorner$
$\Gamma, x\colon S \times T \mid \pi_2 x\colon T$	$z\colon \ulcorner \Gamma, S, T \urcorner \mid \ulcorner \pi_2 \circ \pi_{S \times T} \urcorner z\colon \ulcorner T \urcorner$
$\Gamma, x\colon S \mid \mathsf{inl}\, x\colon S + T$	$z\colon \ulcorner \Gamma, S \urcorner \mid \mathsf{inl} \ulcorner \pi_S \urcorner z\colon \ulcorner S \urcorner + \ulcorner T \urcorner$
$\Gamma, y\colon T \mid \mathsf{inr}\, y\colon S + T$	$z\colon \ulcorner \Gamma, T \urcorner \mid \mathsf{inr} \ulcorner \pi_T \urcorner z\colon \ulcorner S \urcorner + \ulcorner T \urcorner$
$\dfrac{\Gamma, x\colon S \mid s\colon U \qquad \Gamma, y\colon T \mid t\colon U}{\Gamma, z\colon S + T \mid \mathsf{case}\ z\ \mathsf{of}\ x \Rightarrow s,\ y \Rightarrow t\colon U}$	See below
$\dfrac{\Gamma, x\colon S \mid t\colon T \qquad \Gamma \mid s\colon S}{\Gamma \mid t[s/x]\colon T}$	See below

TABLE 5. Term translations rules for $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$.

Notice that for pairing, we use the function symbol $\ulcorner [\![\langle s,\, t\rangle]\!] \urcorner$ (that is, $U[\![\langle s,\, t\rangle]\!]$ in $\mathcal{L}(\mathcal{E})$). Since $[\![\langle s,\, t\rangle]\!]$ is a homomorphism in $\mathcal{E}_{\mathbb{G}}$, $U[\![\langle s,\, t\rangle]\!]$ is an arrow in $\mathcal{E}$. Hence, it makes sense to translate the term $\langle s,\, t\rangle$ this way, because every arrow in $\mathcal{E}$ corresponds to a function symbol in $\mathcal{L}(\mathcal{E})$. Unfortunately, this translation hides the relevant features of the term $\langle s,\, t\rangle$ — namely, that it is a term built by pairing. Also, it is the only translation rule which relies on the semantics of our logic to translate a term of $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$. Nonetheless, this translation or something like it is necessary, to ensure that arbitrary terms of $\mathcal{L}(\mathcal{E})$ cannot be substituted for x (say) in $\ulcorner \langle x,\, y\rangle \urcorner$.

In the translation of the case statement, we use the fact that $\mathcal{E}_{\mathbb{G}}$ is distributive. Thus,

$$T_1 \times \ldots \times T_n \times (S + T) = (T_1 \times \ldots \times T_n \times S) + (T_1 \times \ldots \times T_n \times T).$$

Since U preserves coproducts, we may take $\ulcorner \Gamma, S + T \urcorner = \ulcorner \Gamma, S \urcorner + \ulcorner \Gamma, T \urcorner$. Consequently, we translate a case statement constructed thus

$$\frac{\Gamma, x\colon S \mid s\colon U \qquad \Gamma, y\colon T \mid t\colon U}{\Gamma, z\colon S + T \mid \mathsf{case}\ z\ \mathsf{of}\ x \Rightarrow s,\ y \Rightarrow t\colon U}$$

to a construction

$$\frac{x\colon \ulcorner\Gamma, S\urcorner \,|\, \ulcorner s\urcorner\colon \ulcorner U\urcorner \qquad y\colon \ulcorner\Gamma, T\urcorner \,|\, \ulcorner t\urcorner\colon \ulcorner U\urcorner}{z\colon \ulcorner\Gamma, S\urcorner + \ulcorner\Gamma, T\urcorner \,|\, \mathsf{case}\ z\ \mathsf{of}\ x \Rightarrow \ulcorner s\urcorner,\ y \Rightarrow \ulcorner t\urcorner\colon \ulcorner U\urcorner}$$

We next discuss the translation of a term constructed by substitution. Suppose that $\Gamma, x\colon S \,|\, t\colon T$ and $\Gamma \,|\, s\colon S$. The translation of the former is a term

$$z\colon \ulcorner\Gamma, S\urcorner \,|\, \ulcorner t\urcorner\colon \ulcorner T\urcorner.$$

We will construct a term that allows a substitution for z. Let

$$f = \langle \mathsf{id}_{[\![\Gamma]\!]}, [\![s]\!]\rangle \colon [\![\Gamma]\!] \longrightarrow [\![\Gamma]\!] \times [\![S]\!].$$

Then the translation of f (in the context Γ) is given by

$$y\colon \ulcorner\Gamma\urcorner \,|\, \ulcorner f\urcorner y\colon \ulcorner\Gamma, S\urcorner.$$

Thus, we can now use the substitution constructor in $\mathcal{L}(\mathcal{E})$ to construct

$$\frac{z\colon \ulcorner\Gamma, S\urcorner, y\colon \ulcorner\Gamma\urcorner \,|\, \ulcorner t\urcorner\colon \ulcorner T\urcorner \qquad y\colon \ulcorner\Gamma\urcorner \,|\, \ulcorner f\urcorner y\colon \ulcorner\Gamma, S\urcorner}{y\colon \ulcorner\Gamma\urcorner \,|\, \ulcorner t\urcorner[\ulcorner f\urcorner y/z]\colon \ulcorner T\urcorner} \tag{21}$$

(note the use of weakening in the term $\ulcorner t\urcorner$). We take this term to be $\ulcorner t[s/x]\urcorner$.

By the definition of the translation of terms, it is easy to confirm the following.

THEOREM 4.2.1. *For any term* $\Gamma \,|\, t\colon T$,

$$[\![\ulcorner\Gamma \,|\, t\urcorner]\!] = U[\![\Gamma \,|\, t]\!].$$

PROOF. By induction on the construction of the term. For variables,

$$\ulcorner\Gamma, x\colon T \,|\, x\colon T\urcorner = z\colon \ulcorner\Gamma \times T\urcorner \,|\, \ulcorner\pi_S\urcorner z\colon \ulcorner T\urcorner,$$

and so

$$[\![\ulcorner\Gamma, x \,|\, x\urcorner]\!] = U\pi_S \circ [\![z]\!] = U\pi_S = U[\![\Gamma, x \,|\, x]\!].$$

Other cases are proved similarly, while pairing is trivial. We include the proof for case and substitution.

Given a case term,

$$\Gamma, z\colon S + T \,|\, \mathsf{case}\ z\ \mathsf{of}\ x \Rightarrow s,\ y \Rightarrow t\colon U,$$

its translation is

$$z\colon \ulcorner\Gamma, S\urcorner + \ulcorner\Gamma, T\urcorner \,|\, \mathsf{case}\ z\ \mathsf{of}\ x \Rightarrow \ulcorner s\urcorner,\ y \Rightarrow \ulcorner t\urcorner\colon \ulcorner U\urcorner.$$

This term in $\mathcal{L}(\mathcal{E})$ is interpreted as $[[\![\ulcorner s\urcorner]\!], [\![\ulcorner t\urcorner]\!]] = [U[\![s]\!], U[\![t]\!]] = U[[\![s]\!], [\![t]\!]]$.

As in (21), a substitution $\Gamma \,|\, t[s/x]\colon T$ is translated to the term

$$y\colon \ulcorner\Gamma\urcorner \,|\, \ulcorner t\urcorner[\ulcorner f\urcorner y/z]\colon \ulcorner T\urcorner,$$

where $f = \langle \mathsf{id}_{[\![\Gamma]\!]}, [\![s]\!]\rangle$. Thus, one calculates the interpretation of $\ulcorner t[s/x]\urcorner$ as

$$[\![\ulcorner t\urcorner]\!] \circ \pi_{[\![\ulcorner\Gamma, S\urcorner]\!]} \circ \langle \mathsf{id}_{[\![\ulcorner\Gamma\urcorner]\!]}, [\![\ulcorner f\urcorner y]\!]\rangle = U[\![t]\!] \circ U[\![f]\!] \circ [\![y]\!] = U([\![t]\!] \circ \langle \mathsf{id}_{[\![\Gamma]\!]}, [\![s]\!]\rangle).$$

$\square$

4.2.3. The internal $\square$ operator in $\mathcal{L}(\mathcal{E})$. Before defining the translation of formulas in $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$ to $\mathcal{L}(\mathcal{E})$, we must introduce an internal $\square$ operator. This operator takes formulas φ over one variable z of type $\ulcorner\Gamma\urcorner$ to the largest subcoalgebra of $[\![\varphi]\!]$. Given this operator for unary predicates over coalgebra types, we can then define the $\square$ operator for bisimulations and n-simulations generally, by using the work of Sections 2.5 and 2.7, although these extended operators are not **S4**. We will show that, if $\mathbb{G}$ preserves regular relations, the n-ary $\square$ operator is "almost" **S4** — that is, it will preserve binary meets, but still one does not expect $\square$ to preserve $\top$. In Section 4.2.6, we will show that if the bisimulation $\square$ operator preserves binary meets, then bisimulations compose, using the internal logic.

In order to give this translation, we must first augment the language $\mathcal{L}(\mathcal{E})$ with a modal operator $\square$ representing the "greatest subcoalgebra" construction. Thus, we add the formula formation rule

$$\frac{z\colon \ulcorner\Gamma\urcorner \,|\, \varphi}{z\colon \ulcorner\Gamma\urcorner \,|\, \square\varphi} \tag{22}$$

Notice that this modal operator is only defined for formulas over one variable of type $\ulcorner\Gamma\urcorner$ for some context Γ in $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$. The interpretation of $\square$ is defined by

$$[\![z\colon \ulcorner\Gamma\urcorner \,|\, \square\varphi]\!] = \square_{[\![\Gamma]\!]}[\![\varphi]\!].$$

In other words $[\![\square\varphi]\!]$ is the carrier of the largest subcoalgebra of $[\![\Gamma]\!]$ contained in $[\![\varphi]\!]$.

Theorem 2.2.16 stated that $\square$ is an **S4** modal operator. Consequently, we add the standard **S4** axioms, together with an axiom for substitution of homomorphic terms (justified by Theorem 2.2.17).

($\square$1) $\dfrac{z\colon \ulcorner\Gamma\urcorner \,|\, \varphi \vdash \psi}{z\colon \ulcorner\Gamma\urcorner \,|\, \square\varphi \vdash \square\psi}$

($\square$2) $z\colon \ulcorner\Gamma\urcorner \,|\, \square\varphi \vdash \varphi$

($\square$3) $z\colon \ulcorner\Gamma\urcorner \,|\, \square\varphi \vdash \square\square\varphi$

($\square$4) $z\colon \ulcorner\Gamma\urcorner \,|\, \square\varphi \wedge \square\psi \vdash \square(\varphi \wedge \psi)$

($\square$5) For any term $\Gamma \,|\, t\colon T$ in $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$ and formula $z\colon \ulcorner T\urcorner \,|\, \varphi$,

$$w\colon \ulcorner\Gamma\urcorner \,|\, (\square\varphi(z))[\ulcorner t\urcorner/z] \dashv\vdash \square(\varphi(z)[\ulcorner t\urcorner/z])$$

($\square$6) $z\colon \ulcorner\Gamma\urcorner \,|\, \top \vdash \square\top$

Because $\square$ does not commute with arbitrary substitutions, in general,

$$[\![\square(\varphi[t/x])]\!] \neq (\square\varphi)[t/x].$$

In fact, the formula on the left is defined only if the domain of $[\![t]\!]$ is the carrier of a coalgebra, while the right hand formula requires that the codomain of $[\![t]\!]$ is the carrier of a coalgebra. Hereafter, we will write $\Box\varphi[t/x]$, or just $\Box\varphi(t)$, to denote $[\![\Box(\varphi[t/x])]\!]$. Similarly, when we write

$$x\colon \ulcorner\Gamma\urcorner, \Delta \mid \Box\varphi(x),$$

we mean the formula obtained by weakening the context of $x\colon \ulcorner\Gamma\urcorner \mid \Box\varphi$. Notice that, in general,

$$(\Box\varphi(x))[t/x] \neq \Box\varphi(t).$$

In Section 2.7, we saw that there is a modal operator $\Box$ taking n-ary relations R on $U\langle A_1, \alpha_1\rangle, \ldots, U\langle A_n, \alpha_n\rangle$ to the largest n-simulation contained in R, which is "almost" **S4** (in particular, it is normal, but $\Box$ does not typically preserve $\top$) if $\mathbb{G}$ preserves regular relations. This modal operator is defined in terms of $\exists$, U and the subcoalgebra operator $\Box$. Thus, we can explicitly define the n-simulation $\Box$ operator in our internal logic. To simplify notation, let Γ be the context $x_1\colon T_1, \ldots, x_n\colon T_n$ in $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$ and

$$p = \langle U\pi_1, \ldots, U\pi_n\rangle \colon [\![\ulcorner T_1\urcorner]\!] \times \ldots \times [\![\ulcorner T_n\urcorner]\!] \longrightarrow [\![\ulcorner\Gamma\urcorner]\!].$$

More precisely, we want the interpretation of $x_1\colon \ulcorner T_1\urcorner, \ldots, x_n\colon \ulcorner T_n\urcorner \mid \Box\psi$ to be

$$[\![\Box\psi]\!] = \exists_p \Box_{[\![\ulcorner T_1\urcorner]\!]\times\ldots\times[\![\ulcorner T_n\urcorner]\!]} p^*[\![\psi]\!].$$

Accordingly, we define $\Box\psi$ to denote the formula

$$x_1\colon \ulcorner T_1\urcorner, \ldots, x_n\colon \ulcorner T_n\urcorner \mid \exists_{z\colon \ulcorner\Gamma\urcorner}\Big(\bigwedge_i \ulcorner\pi_i\urcorner z = w_i \wedge \Box\psi(\ulcorner\pi_1\urcorner z, \ldots, \ulcorner\pi_n\urcorner z)\Big) \tag{23}$$

Here, the formula $\Box\psi(\ulcorner\pi_1\urcorner z, \ldots, \ulcorner\pi_n\urcorner z)$ stands for the formula constructed by applying $\Box$ to $\psi(\ulcorner\pi_1\urcorner z, \ldots, \ulcorner\pi_n\urcorner z)$, that is, it denotes

$$z\colon \ulcorner\Gamma\urcorner \mid \Box(\psi[\ulcorner\pi_1\urcorner z/x_1]\ldots[\ulcorner\pi_n\urcorner z/x_n]).$$

Since the $\Box$ operator for variables $z\colon \ulcorner\Gamma\urcorner$ was previously defined, this formula is well-defined. Next, we show that this definition does what it is supposed to.

THEOREM 4.2.2. *For any formula $x_1\colon \ulcorner T_1\urcorner, \ldots, x_n\colon \ulcorner T_n\urcorner \mid \Box\varphi$,*

$$[\![\Box\varphi]\!] = \Box_{[\![T_1]\!],\ldots,[\![T_n]\!]}[\![\varphi]\!]$$

PROOF. One uses the fact that

$$z\colon [\![\Gamma]\!], x_1\colon [\![T_1]\!], \ldots, x_n\colon [\![T_n]\!] \mid \bigwedge_i \ulcorner\pi_i\urcorner z = x_i \dashv\vdash \langle\ulcorner\pi_i\urcorner z, \ldots, \ulcorner\pi_n\urcorner z\rangle = \langle x_1, \ldots, x_n\rangle.$$

By definition of $\Box\varphi$ in (23), we have

$$\begin{aligned}
[\![\Box\varphi]\!] &= [\![\exists_{z\colon \ulcorner\Gamma\urcorner}(\bigwedge_i \ulcorner\pi_i\urcorner z = x_i \wedge \Box\varphi(\ulcorner\pi_1\urcorner z, \dots, \ulcorner\pi_n\urcorner z))]\!] \\
&= [\![\exists_{z\colon \ulcorner\Gamma\urcorner}(\langle\ulcorner\pi_1\urcorner z, \dots, \ulcorner\pi_n\urcorner z\rangle = \langle x_1, \dots, x_n\rangle \wedge \Box\varphi(\ulcorner\pi_1\urcorner z, \dots, \ulcorner\pi_n\urcorner z))]\!] \\
&= \exists_{[\![\langle\ulcorner\pi_1\urcorner z, \dots, \ulcorner\pi_n\urcorner z\rangle]\!]}[\![\Box\varphi(\ulcorner\pi_1\urcorner z, \dots, \ulcorner\pi_n\urcorner z)]\!] \\
&= \exists_p \Box[\![\varphi(\ulcorner\pi_1\urcorner z, \dots, \ulcorner\pi_n\urcorner z)]\!] \\
&= \exists_p \Box p^*[\![\varphi]\!] = \Box[\![\varphi]\!].
\end{aligned}$$

$\Box$

The axioms ($\Box$1) - ($\Box$ 5) generalize to axioms of the analogous formulas for the n-simulation $\Box$ operator — assuming that the comonad $\mathbb{G}$ preserves regular relations for the normality axiom. Some of these axioms are easily provable in the internal logic (for instance, the deflationary axiom $\Box\varphi \vdash \varphi$), and perhaps all of them are provable with sufficient work (although it appears the normality axiom would require some semantic argument to use the assumption that $\mathbb{G}$ preserves pullbacks). Nonetheless, we rely on the work of Sections 2.5 and 2.7 to justify each of the following, which we take to be axioms.

Notably, axiom ($\Box$6) does *not* typically hold for the n-ary $\Box$ operator. Instead, $\Box\top$ is properly contained in $\top$ — that is, the largest bisimulation of $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ is typically a proper subobject of $A \times B$.

Let $\Gamma = y_1\colon T_1, \dots, y_n\colon T_n$ (in $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$) and $\Delta = x_1\colon \ulcorner T_1\urcorner, \dots, x_n\colon \ulcorner T_n\urcorner$ (in $\mathcal{L}(\mathcal{E})$).

($\Box 1'$) $\dfrac{\Delta \mid \varphi \vdash \psi}{\Delta \mid \Box\varphi \vdash \Box\psi}$

($\Box 2'$) $\Delta \mid \Box\varphi \vdash \varphi$

($\Box 3'$) $\Delta \mid \Box\varphi \vdash \Box\Box\varphi$

($\Box 4'$) (If $\mathbb{G}$ preserves regular relations) $\Delta \mid \Box\varphi \wedge \Box\psi \vdash \Box(\varphi \wedge \psi)$

($\Box 5'$) For any term $y\colon S \mid t\colon T$ in $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$ and formula $\Delta, x\colon \ulcorner T\urcorner \mid \varphi$,

$$\Delta, y\colon \ulcorner S\urcorner \mid (\Box\varphi)[\ulcorner t\urcorner/x] \dashv\vdash \Box(\varphi[\ulcorner t\urcorner/x])$$

That the axioms ($\Box$1) - ($\Box$3) are sound follows from the fact that (the interpretation of) $\Box$ is a comonad, the soundness of ($\Box$4) was proved in Corollary 2.5.26, and of ($\Box$5) follows from Theorem 2.5.19, in which we proved that $\Box$ commutes with pullback along products of homomorphisms.

The next theorem is provably equivalent to Theorem 2.7.8, in which we showed that $\Box\pi^* \leq \pi^*\Box$. The following theorem can be understood as stating that, if R is an n-simulation, then $\exists_\pi R$ is also an m-simulation (where π is a projection — and for suitable m). The statement of the theorem is an internal version of this claim.

THEOREM 4.2.3. *If $\mathbb{G}$ preserves regular relations, then for any formula φ over context $x\colon \ulcorner S \urcorner, y_1\colon \ulcorner T_1 \urcorner, \ldots, y_n\colon \ulcorner T_n \urcorner$,*

$$\models \exists_{x\colon S}\Box\varphi \vdash \Box\exists_{x\colon S}\varphi.$$

PROOF. Let π denote the projection $[\![\ulcorner S \urcorner]\!] \times \prod [\![\ulcorner T_i \urcorner]\!] \rightarrow \prod [\![\ulcorner T_i \urcorner]\!]$. Then, by Theorem 2.7.8,

$$\Box[\![\varphi]\!] \leq \Box\pi^*\exists_\pi[\![\varphi]\!] \leq \pi^*\Box\exists_\pi[\![\varphi]\!]$$

and so $\exists_\pi\Box[\![\varphi]\!] \leq \Box\exists_\pi\varphi$. Thus, $\exists_{x\colon S}\Box\varphi \vdash \Box\exists_{x\colon S}\varphi$. $\Box$

Thus, in case $\mathbb{G}$ preserves regular relations, then we may add another $\Box$ axiom:

($\Box 6'$) $\models \exists_{x\colon S}\Box\varphi \vdash \Box\exists_{x\colon S}\varphi$.

4.2.4. Translation of formulas. We next give a translation of formulas in $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$ to formulas in $\mathcal{L}(\mathcal{E})$. The inductive definition of the translation is given in Table 6.

Formula formation rule	Translation
$x\colon T \mid \varphi_P(x)$	$x\colon \ulcorner T \urcorner \mid \varphi_P(x)$
$x\colon T, y\colon T \mid x = y$	$z\colon \ulcorner T \times T \urcorner \mid \Box(\ulcorner \pi_1 \urcorner z = \ulcorner \pi_2 \urcorner z)$
$\Gamma \mid \top$	$z\colon \ulcorner \Gamma \urcorner \mid \top$
$\Gamma \mid \bot$	$z\colon \ulcorner \Gamma \urcorner \mid \bot$
$\Gamma \mid \varphi \wedge \psi$	$z\colon \ulcorner \Gamma \urcorner \mid \ulcorner \varphi \urcorner \wedge \ulcorner \psi \urcorner$
$\Gamma \mid \varphi \rightarrow \psi$	$z\colon \ulcorner \Gamma \urcorner \mid \Box(\ulcorner \varphi \urcorner \rightarrow \ulcorner \psi \urcorner)$
$\Gamma \mid \exists_{x\colon T}\varphi$	$z\colon \ulcorner \Gamma \urcorner \mid \exists_{x\colon \ulcorner \Gamma, T \urcorner}(\ulcorner \pi_\Gamma \urcorner(x) = z \wedge \ulcorner \varphi \urcorner(x))$
$\Gamma \mid \forall_{x\colon T}\varphi$	$z\colon \ulcorner \Gamma \urcorner \mid \Box\forall_{x\colon \ulcorner \Gamma, T \urcorner}(\ulcorner \pi_\Gamma \urcorner x = z \rightarrow \ulcorner \varphi \urcorner(x))$
$\Gamma \mid \varphi[t/x]$	$z\colon \ulcorner \Gamma \urcorner \mid \ulcorner \varphi \urcorner[\ulcorner \langle x_1, \ldots, x_n, t \rangle \urcorner / w]$
$\Gamma, \Delta \mid \varphi$	$z\colon \ulcorner \Gamma, \Delta \urcorner \mid \ulcorner \varphi \urcorner[\ulcorner \pi_\Gamma \urcorner z / z]$

TABLE 6. Formula formation and interpretation

REMARK 4.2.4. One could add a closure operator ∇ to the language of $\mathcal{E}_{\mathbb{G}}$ as well, where the interpretation comes from the closure operator in Section 2.7. The standard axioms for closure operators would apply, as well as the following axiom (taking its premise from $\mathcal{L}(\mathcal{E})$, its conclusion is in $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$):

$$\frac{x\colon \ulcorner T_1 \urcorner \times \ldots \times \ulcorner T_n \urcorner \mid \exists_{z\colon \ulcorner \Gamma \urcorner}(pz = x \wedge \ulcorner \psi \urcorner(z)) \vdash \exists_{z\colon \ulcorner \Gamma \urcorner}(pz = x \wedge \ulcorner \varphi \urcorner(z))}{\Gamma \mid \psi \vdash \nabla\varphi}$$

We translate the formula $\Gamma \mid \nabla\varphi$ into $\mathcal{L}(\mathcal{E})$ as

$$z\colon \ulcorner \Gamma \urcorner \mid \Box(\exists_{y\colon \ulcorner \Gamma \urcorner}(\bigwedge_i \ulcorner \pi_i \urcorner y = \ulcorner \pi_i \urcorner z \wedge \ulcorner \varphi \urcorner(y))).$$

One can show (as an extension of the following theorem) that

$$[\![\ulcorner \nabla\varphi \urcorner]\!] = U[\![\nabla\varphi]\!],$$

but we omit the details.

THEOREM 4.2.5. *For every formula* $\Gamma \,|\, \varphi$ *of* $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$,

$$[\![\ulcorner \Gamma \,|\, \varphi \urcorner]\!] = U[\![\Gamma \,|\, \varphi]\!].$$

PROOF. As in the proof of Theorem 4.2.1, we prove the result by induction, but present only a few of the cases here.

For formulas of the form $\Gamma \,|\, \varphi \to \psi$, we see that

$$\begin{aligned} [\![\varphi \to \psi]\!] &= [\![\varphi]\!] \to [\![\psi]\!] \\ &= [U[\![\varphi]\!] \to U[\![\psi]\!]] && \text{(by Theorem 2.2.24)} \\ &= [[\![\ulcorner\varphi\urcorner]\!] \to [\![\ulcorner\psi\urcorner]\!]] && \text{(by inductive hypothesis)} \end{aligned}$$

On the other hand,

$$\begin{aligned} [\![\ulcorner \varphi \to \psi \urcorner]\!] &= [\![\Box(\ulcorner\varphi\urcorner \to \ulcorner\psi\urcorner)]\!] \\ &= \Box[\![(\ulcorner\varphi\urcorner \to \ulcorner\psi\urcorner)]\!] \\ &= \Box([\![\ulcorner\varphi\urcorner]\!] \to [\![\ulcorner\psi\urcorner]\!]) = U[[\![\ulcorner\varphi\urcorner]\!] \to [\![\ulcorner\psi\urcorner]\!]]. \end{aligned}$$

We next consider formulas of the form $\Gamma \,|\, \forall_{x\colon T}\varphi$. We use the fact that, in any logos, for any f,

$$[\![\forall_{x\colon T}(f(x) = y \to \varphi(x,y))]\!] = \forall_f [\![\varphi(x,y)]\!],$$

and also that, for any homomorphism f, $\forall_f = [-] \circ \forall_{Uf} \circ U$ (Theorem 4.1.10). Consequently,

$$U[\![\forall_{x\colon T}\varphi]\!] = U(\forall_{\pi_\Gamma}[\![\varphi]\!]) = \Box\forall_{U\pi_\Gamma}[\![\ulcorner\varphi\urcorner]\!] = [\![\Box\forall_{x\colon \ulcorner\Gamma, T\urcorner}(\ulcorner\pi_\Gamma\urcorner x = z \to \ulcorner\varphi\urcorner)]\!].$$

We translate substitutions

$$\frac{\Gamma, x\colon T \,|\, \varphi \qquad \Gamma \,|\, t\colon T}{\Gamma \,|\, \varphi[t/x]}$$

to substitutions

$$\frac{\dfrac{w\colon \ulcorner\Gamma, T\urcorner \,|\, \ulcorner\varphi\urcorner}{w\colon \ulcorner\Gamma, T\urcorner, z\colon \ulcorner\Gamma\urcorner \,|\, \ulcorner\varphi\urcorner} \qquad \dfrac{z\colon \ulcorner\Gamma\urcorner \,|\, \ulcorner t\urcorner\colon \ulcorner T\urcorner}{z\colon \ulcorner\Gamma\urcorner \,|\, \ulcorner\langle x_1, \ldots, x_n, t\rangle\urcorner\colon \ulcorner\Gamma, T\urcorner}}{z\colon \ulcorner\Gamma\urcorner \,|\, \varphi[\ulcorner\langle x_1, \ldots, x_n, t\rangle\urcorner/w]}$$

The interpretation of $\ulcorner\varphi[t/x]\urcorner$, then, is calculated as

$$\begin{aligned} [\![\ulcorner\varphi[t/x]\urcorner]\!] &= [\![\ulcorner\varphi\urcorner[\ulcorner\langle x_1, \ldots, x_n, t\rangle\urcorner/w]]\!] \\ &= [\![\ulcorner\langle x_1, \ldots, x_n, t\rangle\urcorner]\!]^*[\![\ulcorner\varphi\urcorner]\!] \\ &= (U[\![\langle x_1, \ldots, x_n, t\rangle]\!])^*(U[\![\varphi]\!]) \\ &= U(\langle 1, [\![t]\!]\rangle^*[\![\varphi]\!]) = U[\![\varphi[t/x]]\!]. \end{aligned}$$

□

Thus, we have constructed a translation from the language $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$ to the language $\mathcal{L}(\mathcal{E})$ which takes a formula $\Gamma \,|\, \varphi$ to a formula of $\mathcal{L}(\mathcal{E})$ which is interpreted as the carrier of $[\![\varphi]\!]$. This translation is "truth-preserving" in the sense that $\varphi \vdash \psi$ is valid in $\mathcal{E}_{\mathbb{G}}$ just in case $\ulcorner\varphi\urcorner \vdash \ulcorner\psi\urcorner$ is valid in $\mathcal{E}$.

THEOREM 4.2.6. *For every sequent* $\Gamma \,|\, \varphi \vdash \psi$ *in* $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$, $\models \ulcorner\Gamma\urcorner \,|\, \ulcorner\varphi\urcorner \vdash \ulcorner\psi\urcorner$ *iff* $\models \Gamma \,|\, \varphi \vdash \psi$.

PROOF. Suppose $\models \Gamma \,|\, \varphi \vdash \psi$. Then

$$[\![\ulcorner\Gamma\urcorner \,|\, \ulcorner\varphi\urcorner]\!] = U[\![\Gamma \,|\, \varphi]\!] \leq U[\![(]\!]\Gamma \,|\, \psi) = [\![\ulcorner\Gamma\urcorner \,|\, \ulcorner\psi\urcorner]\!].$$

Conversely, suppose $\models \ulcorner\Gamma\urcorner \,|\, \ulcorner\varphi\urcorner \vdash \ulcorner\psi\urcorner$. Then

$$U[\![\Gamma \,|\, \varphi]\!] = [\![\ulcorner\Gamma\urcorner \,|\, \ulcorner\varphi\urcorner]\!] \leq [\![\ulcorner\Gamma\urcorner \,|\, \ulcorner\psi\urcorner]\!] = U[\![\Gamma \,|\, \psi]\!].$$

Corollary 1.2.10 completes the proof. □

Hence, we may add a rule of inference

$$\text{(Tr1)} \quad \frac{\Gamma \,|\, \varphi \vdash \psi}{\ulcorner\Gamma\urcorner \,|\, \ulcorner\varphi\urcorner \vdash \ulcorner\psi\urcorner}$$

4.2.5. Coinduction. We will use the internal logic to prove a formula intended to represent the principle of coinduction. The principle of coinduction for the final coalgebra $H1$ states that the largest coalgebraic relation on $H1$ is equality, i.e.,

$$\Delta_{H1} = H1.$$

Equivalently, coinduction says that the largest bisimulation $\Box_{H1,H1}\top$ is just Δ_{UH1} (see Theorem 2.6.3). In what follows, recall that the type 1 in $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$ is interpreted as $H1$, the final coalgebra, so that $\ulcorner 1\urcorner$ is interpreted as $UH1$. Thus, we wish to prove the formula

$$x\colon \ulcorner 1\urcorner, y\colon \ulcorner 1\urcorner \,|\, \Box\top(x, y) \vdash x = y.$$

By definition of $\Box$ (in the formula (23)), this is the formula

$$\text{(24)} \qquad x\colon \ulcorner 1\urcorner, y\colon \ulcorner 1\urcorner \,|\, \exists_{z\colon \ulcorner 1\times 1\urcorner}(\ulcorner\pi_1\urcorner z = x \wedge \ulcorner\pi_2\urcorner z = y \wedge \Box\top) \vdash x = y.$$

Since $\Box\top \vdash \top$, it is sufficient to prove the simpler formula

$$\text{(25)} \qquad x\colon \ulcorner 1\urcorner, y\colon \ulcorner 1\urcorner \,|\, \exists_{z\colon \ulcorner 1\times 1\urcorner}(\ulcorner\pi_1\urcorner z = x \wedge \ulcorner\pi_2\urcorner z = y) \vdash x = y.$$

Indeed, since $\top \vdash \Box\top$ (for the unary $\Box$ which appears here), (24) and (25) are equivalent.

In $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$, the formula

$$x\colon 1, y\colon 1 \,|\, \top \vdash x = y$$

is provable, using (Pr1) and the equality axioms. Hence, by Theorem 4.2.6, its translation

$$z\colon \ulcorner 1 \times 1 \urcorner \mid \top \vdash \Box(\ulcorner \pi_1 \urcorner z = \ulcorner \pi_2 \urcorner z)$$

holds in $\mathcal{L}(\mathcal{E})$. Again, we apply the deflationary axiom ($\Box$2) to conclude

$$z\colon \ulcorner 1 \times 1 \urcorner \mid \top \vdash \ulcorner \pi_1 \urcorner z = \ulcorner \pi_2 \urcorner z.$$

From this, it is a simple exercise to prove (25).

We may prove a stronger claim regarding coinduction. From Theorem 1.5.25, we know that, if $\mathbb{G}$ preserves weak pullbacks, then a coalgebra $\langle A,\, \alpha\rangle$satisfies coinduction iff $\langle A,\, \alpha\rangle$is simple. From (the dual of) Theorem 1.5.14, we know that, if $\mathbb{G}$ preserves weak pullbacks, then $\langle A,\, \alpha\rangle$is simple iff $!\colon \langle A,\, \alpha\rangle \twoheadrightarrow H1$ is a regular mono. We will now present this connection in the internal logic.

By assumption, $\mathcal{E}$ has regular epi-regular mono factorizations, so every mono is a regular mono. Thus, we can represent the claim that ! is a regular mono by the familiar sequent

$$x\colon \ulcorner T \urcorner, y\colon \ulcorner T \urcorner \mid \ulcorner ! \urcorner x = \ulcorner ! \urcorner y \vdash x = y. \tag{26}$$

Here, we use the fact that U preserves regular monos, and so ! is a regular mono in $\mathcal{E}_{\mathbb{G}}$ iff ! is a mono in $\mathcal{E}$. Hence, the theorem we wish to prove is that (26) is equivalent to (25) (replacing 1 with T in the latter). We do this by proving the antecedents are equivalent. This has the advantage that it makes clear that two elements are mapped via ! to the same element of $UH1$ just in case the elements are bisimilar.

First, we must see how to represent the assumption that $\mathbb{G}$ preserves weak pullbacks. We use it in the proof by applying Theorem 2.5.7, to conclude that the pullback of two $\mathbb{G}$-homomorphisms is a bisimulation. This fact suggests the following internal formula for each term s, t in $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$

$$x\colon \ulcorner S \urcorner, y\colon \ulcorner T \urcorner \mid \ulcorner s \urcorner x = \ulcorner t \urcorner y \vdash \Box(\ulcorner s \urcorner x = \ulcorner t \urcorner y). \tag{27}$$

We treat this formula as an axiom in the following proof.

THEOREM 4.2.7. *Suppose that $\mathbb{G}$ preserves weak pullbacks. For any type T in $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$,*

$$x\colon \ulcorner T \urcorner, y\colon \ulcorner T \urcorner \mid \ulcorner ! \urcorner x = \ulcorner ! \urcorner y \dashv\vdash \exists_{z\colon \ulcorner T \times T \urcorner}(\ulcorner \pi_1 \urcorner z = x \wedge \ulcorner \pi_2 \urcorner z = y).$$

PROOF. By (27), we have

$$x\colon \ulcorner T \urcorner, y\colon \ulcorner T \urcorner \mid \ulcorner ! \urcorner x = \ulcorner ! \urcorner y \vdash \Box(\ulcorner ! \urcorner x = \ulcorner ! \urcorner y).$$

By definition of $\Box$, the consequent is the formula

$$x\colon \ulcorner T \urcorner, y\colon \ulcorner T \urcorner \mid \exists_{z\colon \ulcorner T \times T \urcorner}(\ulcorner \pi_1 \urcorner z = x \wedge \ulcorner \pi_2 \urcorner z = y \wedge \Box(\ulcorner ! \pi_1 \urcorner z = \ulcorner ! \pi_2 \urcorner z)) \tag{28}$$

In the language of $\mathcal{E}_{\mathbb{G}}$, one can prove that

$$z\colon T \times T \mid \top \vdash *(\pi_1 z) = *(\pi_2 z), \tag{29}$$

and $\Box(\ulcorner !\pi_1 \urcorner z = \ulcorner !\pi_2 \urcorner z)$ is the translation of that consequent. Hence, (28) is provably equivalent to

$$x\colon \ulcorner T \urcorner, y\colon \ulcorner T \urcorner \mid \exists_{z\colon \ulcorner T\times T \urcorner}(\ulcorner \pi_1 \urcorner z = x \wedge \ulcorner \pi_2 \urcorner z = y). \tag{30}$$

Hence, we have shown

$$x\colon \ulcorner T \urcorner, y\colon \ulcorner T \urcorner \mid \ulcorner ! \urcorner x = \ulcorner ! \urcorner y \vdash \exists_{z\colon \ulcorner T\times T \urcorner}(\ulcorner \pi_1 \urcorner z = x \wedge \ulcorner \pi_2 \urcorner = y).$$

For the other direction, we again translate (29) and apply ($\Box$2) ($\Box$ is deflationary) to yield the sequent

$$z\colon \ulcorner T \times T \urcorner \mid \ulcorner !\pi_1 \urcorner z = \ulcorner !\pi_2 \urcorner z \vdash .$$

Using the axioms for equality, we get

$$z\colon \ulcorner T \times T \urcorner, x\colon \ulcorner T \urcorner, y\colon \ulcorner T \urcorner \mid \ulcorner \pi_1 \urcorner z = x \wedge \ulcorner \pi_2 \urcorner z = y \vdash \ulcorner ! \urcorner x = \ulcorner ! \urcorner y.$$

An application of existential introduction (i.e., (Str6)) completes the proof. $\Box$

COROLLARY 4.2.8. *Let $\mathbb{G}$ preserve weak pullbacks. Then, for any type T,*

$$\models x\colon \ulcorner T \urcorner, y\colon \ulcorner T \urcorner \mid \ulcorner ! \urcorner x = \ulcorner ! \urcorner y \vdash x = y$$

just in case

$$\models x\colon \ulcorner T \urcorner, y\colon \ulcorner T \urcorner \mid \exists_{z\colon \ulcorner T\times T \urcorner}(\ulcorner \pi_1 \urcorner z = x \wedge \ulcorner \pi_2 \urcorner z = y) \vdash x = y.$$

In other words, the internal principle of coinduction is valid just in case $\ulcorner ! \urcorner$ is monic.

4.2.6. Composition of bisimulations. We offer an example of the internal logic at work in the following theorem, in which we prove that, if $\mathbb{G}$ preserves regular relations, then the composition of two bisimulations is again a bisimulation. From [**JR97**], one finds the well-known theorem that, if $\mathcal{E}$ satisfies the axiom of choice, then bisimulations compose. These two theorems suffice to prove that bisimulations compose in a variety of familiar settings, but in both theorems, the category of bisimulations consists of relations which come with a structure map (as opposed to bisimulations in the sense of Definition 2.5.4, which is more general). We know of no results for categories in which bisimulations are not "merely" relations which come with a structure map.

THEOREM 4.2.9. *If $\mathbb{G}$ preserves regular relations, then bisimulations compose. That is, for any $x\colon \ulcorner R \urcorner, y\colon \ulcorner S \urcorner \mid \varphi$ and $y\colon \ulcorner S \urcorner, z\colon \ulcorner T \urcorner \mid \psi$ such that*

$$x\colon \ulcorner R \urcorner, y\colon \ulcorner S \urcorner \mid \varphi \vdash \Box\varphi \text{ and } y\colon \ulcorner S \urcorner, z\colon \ulcorner T \urcorner \mid \psi \vdash \Box\psi,$$

we have $x\colon \ulcorner R \urcorner, z\colon \ulcorner R \urcorner \mid \psi \circ \varphi \vdash \Box(\psi \circ \varphi)$.

PROOF. By the assumption, one infers that $\psi \wedge \varphi \vdash \Box\psi \wedge \Box\varphi$. Hence, $\psi \wedge \varphi \vdash \Box(\psi \wedge \varphi)$. Applying the cut rule to this sequent and to the sequent

$$\Box(\psi \wedge \varphi) \vdash \exists_{y:\,S}\Box(\psi \wedge \varphi)$$

yields $\psi \wedge \varphi \vdash \exists_{y:\,S}\Box(\psi \wedge \varphi)$ and thus $\exists_{y:\,S}(\psi \wedge \varphi) \vdash \exists_{y:\,S}\Box(\psi \wedge \varphi)$. By Theorem 4.2.3, $\exists_{y:\,S}\Box(\psi \wedge \varphi) \vdash \Box\exists_{y:\,S}(\psi \wedge \varphi)$, and so $\exists_{y:\,S}(\psi \wedge \varphi) \vdash \Box\exists_{y:\,S}(\psi \wedge \varphi)$, as desired. □

REMARK 4.2.10. This proof really doesn't require all of the assumptions on $\mathcal{E}_{\mathbb{G}}$ that we've made in this chapter. Rather, it holds whenever $\mathcal{E}$ has epi-regular mono factorizations, is cocomplete and finitely complete and $\mathbb{G}$ preserves regular relations. In fact, $\mathbb{G}$ may be an endofunctor, rather than a comonad. Thus, it holds under the same assumptions that applied in Section 2.5.2, where we discussed relation-preserving functors.

4.3. A Kripke-Joyal style semantics

Throughout this section, we continue with the assumptions from Section 4.2. That is, we assume that $\mathcal{E}$ is an extensive, well-powered, locally complete logos with all coproducts and regular epi-regular mono factorizations. We also assume that $\mathbb{G}$ nearly preserves pullbacks.

One of the motivations for considering the internal logics $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$ and $\mathcal{L}(\mathcal{E})$ is that, given an injective object C, a coequation over C is just a regular subobject of UHC, where H is right adjoint to $U:\mathcal{E}_{\mathbb{G}} \twoheadrightarrow \mathcal{E}$. In other words, a coequation over C is the interpretation of some formula $x:\ulcorner HC \urcorner \mid \varphi$.

A coalgebra $\langle A, \alpha\rangle$ forces the coequation φ just in case, for every $p:\langle A, \alpha\rangle \twoheadrightarrow HC$, the image of p (more precisely, Up) is contained in the interpretation of φ. Thus, $\langle A, \alpha\rangle \Vdash \varphi$ just in case, for every element p of HC centered at $\langle A, \alpha\rangle$, we have $\mathsf{Im}(p) \leq [\![\varphi]\!]$. This suggests that the standard Kripke-Joyal semantics can be used to express coequation satisfaction in a simple, familiar way.

In this section, we first introduce Kripke-Joyal semantics for a locally complete logos and state (without proof) the Kripke-Joyal semantics theorem. We adopt this semantics for the category $\mathcal{L}(\mathcal{E})$, in which certain formulas represent coequations. Namely, those formulas $\ulcorner\Gamma\urcorner \mid \varphi$ are interpreted as subobjects of $[\![\ulcorner\Gamma\urcorner]\!] = U[\![\Gamma]\!]$, and hence as conditional coequations over $[\![\Gamma]\!]$. We complete this section by proving a couple of theorems about coequation forcing in terms of the internal logic, first introducing an internal version of the **S4** modal operator $\boxtimes$ from Section 3.8.

Let Γ be a context in $\mathcal{L}(\mathcal{E})$ and φ a formula in context Γ. Let A be given and a an element of $[\![\Gamma]\!]$, centered at A, i.e.,

$$a:A \longrightarrow [\![\Gamma]\!].$$

Then we say that $A \Vdash \varphi(a)$ just in case $\mathsf{Im}(a) \le [\![\varphi]\!]$, i.e., $pa \in_A [\![\varphi]\!]$.

The following theorem (which can be found, essentially, in [**LS86, Bor94**], etc.) can be proved in any locally complete logos. As it is a well-known theorem (the Kripke-Joyal theorem — sometimes called Beth-Kripke-Joyal), we omit the proof but include the statement for completeness.

THEOREM 4.3.1. *Let $\Gamma \,|\, \varphi$ be given and $p \in_A [\![\Gamma]\!]$. In each of the clauses below, the context of the formula is Γ, unless stated otherwise.*

(1) $A \Vdash \top(p)$ always.
(2) $A \Vdash \bot(p)$ iff $A = 0$.
(3) $A \Vdash (\varphi \wedge \psi)(p)$ iff $A \Vdash \varphi(p)$ and $A \Vdash \psi(p)$.
(4) $A \Vdash (\varphi \to \psi)(p)$ iff, for every $b \in_B A$ such that $B \Vdash \varphi(pb)$, then also $B \Vdash \psi(pb)$.
(5) $A \Vdash \exists_{x:T}\varphi(x,p)$ iff there is a regular epi $b : B \twoheadrightarrow A$ and a $c \in_B [\![T]\!]$ such that $B \Vdash \varphi(c, pb)$.
(6) $A \Vdash \forall_{x:T}\varphi(x,p)$ iff, for all $b \in_B A$ and $c \in_B [\![T]\!]$, $B \Vdash \varphi(c, pb)$.
(7) $A \Vdash \varphi[t/x](a)$ iff $A \Vdash \varphi([\![t]\!]a)$.
(8) (Weakening) $A \Vdash \Delta, \Gamma \,|\, \varphi(a)$ iff $A \Vdash \Gamma \,|\, \varphi(\pi_{[\![\Gamma]\!]}a)$.

Let $\ulcorner\Gamma\urcorner \,|\, \varphi$ be given. The $[\![\varphi]\!]$ is a conditional coequation over $[\![\ulcorner\Gamma\urcorner]\!]$. A coalgebra $\langle A,\, \alpha\rangle$forces $[\![\varphi]\!]$ just in case, for every homomorphism $p : \langle A,\, \alpha\rangle \twoheadrightarrow [\![\Gamma]\!]$, we have $\mathsf{Im}(Up) \le [\![\varphi]\!]$, equivalently, $\mathsf{Im}(p) \le [\![\Box\varphi]\!]$.

THEOREM 4.3.2. *Let $\ulcorner\Gamma\urcorner \,|\, \varphi$ and $\langle A,\, \alpha\rangle$ be given. Then $\langle A,\, \alpha\rangle \Vdash_{[\![\Gamma]\!]} [\![\varphi]\!]$ (over $[\![\Gamma]\!]$) just in case, for every element $p \in_{\langle A,\, \alpha\rangle} [\![\Gamma]\!]$, that is, every homomorphism $p : \langle A,\, \alpha\rangle \twoheadrightarrow [\![\Gamma]\!]$,*

$$A \Vdash \varphi(Up).$$

We next show that a coalgebra forces φ at Up just in case it forces $\Box\varphi$ at Up. This is an easy corollary to Corollary 2.2.9. It also implies that the quasi-covariety defined by φ is the same as that defined by $\Box\varphi$, i.e., Theorem 3.8.10. In Theorem 4.3.4, we present a similar theorem for the $\boxtimes$ operator.

THEOREM 4.3.3. *Let $\ulcorner\Gamma\urcorner \,|\, \varphi$ and $\langle A,\, \alpha\rangle$ be given, and $p : \langle A,\, \alpha\rangle \twoheadrightarrow [\![\Gamma]\!]$ a homomorphism. Then $A \Vdash \varphi(Up)$ iff $A \Vdash \Box\varphi(Up)$.*

PROOF. Corollary 2.2.9 states that Up factors through $[\![\varphi]\!]$ just in case p factors through $[[\![\varphi]\!]]$ (the largest subcoalgebra of $[\![\varphi]\!]$). Hence, Up factors through $[\![\varphi]\!]$ iff it factors through $\Box[\![\varphi]\!] = [\![\Box\varphi]\!]$ (see Corollary 1.2.10). $\Box$

For the next theorems, we augment the language $\mathcal{L}(\mathcal{E})$ with another **S4** modal operator, $\boxtimes$. The interpretation of $\boxtimes\varphi$ is $\boxtimes[\![\varphi]\!]$, that is, the largest invariant subcoalgebra of $[\![\varphi]\!]$ (see Section 3.8).

THEOREM 4.3.4. *$\langle A, \alpha\rangle \Vdash \varphi(p)$, for every $p \in_{\langle A, \alpha\rangle} [\![\Gamma]\!]$, just in case we also have $\langle A, \alpha\rangle \Vdash \boxtimes\varphi(p)$ for every $p \in_{\langle A, \alpha\rangle} [\![\Gamma]\!]$.*

PROOF. This is just a restatement of Theorem 3.8.9 in terms of the Kripke-Joyal semantics. □

THEOREM 4.3.5. *Let $\ulcorner\Gamma\urcorner \mid \varphi$ be given. If $[\![\Gamma]\!]$ is injective (say, $\Gamma = x\colon HC$), and $p \in_{\langle A, \alpha\rangle} [\![\Gamma]\!]$, then*

$$A \Vdash \boxtimes\varphi(p)$$

iff, for every homomorphism $g\colon[\![\Gamma]\!] \to [\![\Gamma]\!]$, $A \Vdash \varphi(gp)$.

PROOF. Let $A \Vdash \boxtimes\varphi(p)$, where p is a homomorphism, and $g\colon[\![\Gamma]\!] \to [\![\Gamma]\!]$ be given. Then, by definition of $\boxtimes$, $\exists_g[\![\boxtimes\varphi]\!] \leq [\![\varphi]\!]$ and thus,

$$\mathsf{Im}(gp) = \exists_g \mathsf{Im}\, p \leq [\![\varphi]\!].$$

Conversely, suppose that for every such g, $A \Vdash \varphi(gp)$. Then, for every homomorphism $g\colon[\![\Gamma]\!] \to [\![\Gamma]\!]$, we have $\exists_g \mathsf{Im}(p) \leq [\![\varphi]\!]$. But $\boxtimes[\![\varphi]\!]$ was defined to be the join of all those subobjects K of $U[\![\Gamma]\!]$ such that, for every homomorphism $g\colon[\![\Gamma]\!] \to [\![\Gamma]\!]$, we have $\exists_g K \leq [\![\varphi]\!]$ (see Section 3.8). Hence, the result follows. □

4.4. Pointwise forcing of coequations

Again, throughout this section, $\mathcal{E}$ is an extensive, well-powered, locally complete logos with all coproducts and regular epi-regular mono factorizations and that $\mathbb{G}\colon\mathcal{E} \to \mathcal{E}$ is a comonad that nearly preserves pullbacks.

Let Γ be a context in $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$, and φ a formula over $\ulcorner\Gamma\urcorner$, so that $[\![\varphi]\!]$ is a conditional coequation over $[\![\Gamma]\!]$. As we saw in the previous section, a coalgebra $\langle A, \alpha\rangle$ forces φ just in case, for every element p of $[\![\Gamma]\!]$ centered at $\langle A, \alpha\rangle$, $A \Vdash \varphi(Up)$. In other words, the Kripke-Joyal semantics give a means of stating that $\langle A, \alpha\rangle$ forces a coequation under a particular coloring, where $\langle A, \alpha\rangle$ forces the coequation (with no qualifications) if it forces it under every coloring.

Alternatively, we could consider the elements of A and ask which elements satisfy φ. That is, which elements are mapped into $[\![\varphi]\!]$ under every mapping $p\colon\langle A, \alpha\rangle \to [\![\Gamma]\!]$? Clearly, $\langle A, \alpha\rangle$ iff for all $p\colon\langle A, \alpha\rangle \to [\![\Gamma]\!]$ and all $a \in A$, $pa \in [\![\varphi]\!]$. In Section 4.3, we stripped away the quantifier ranging over colorings and defined "$\langle A, \alpha\rangle$ forces φ under p." In this section, we strip away the quantifier ranging over elements of A and define "$\langle A, \alpha\rangle$ forces φ at a," where a is an element of A (i.e., $a \in_B A$ for some $B \in \mathcal{E}$).

DEFINITION 4.4.1. Let $\ulcorner\Gamma\urcorner \mid \varphi$ and $\langle A, \alpha\rangle$ be given, with $a \in A$ (i.e., $a\colon\bullet \to A$ in $\mathcal{E}$). Then we say

$$\langle A, \alpha\rangle \Vdash \varphi[a]$$

iff, for every homomorphism $p{:}\langle A, \alpha\rangle \twoheadrightarrow [\![\Gamma]\!]$, we have $\mathsf{Im}(pa) \leq [\![\varphi]\!]$.

We use the square brackets for pointwise forcing to distinguish the notation from $A \Vdash \varphi(p)$, where $p \in_{\langle A, \alpha\rangle} [\![\Gamma]\!]$. Clearly, $\langle A, \alpha\rangle \Vdash \varphi[a]$ iff, for every $p \in_{\langle A, \alpha\rangle} [\![\Gamma]\!]$,

$$B \Vdash \varphi(pa)$$

(where B is the domain of a, i.e., $a \in_B A$).

THEOREM 4.4.2. *Let $\ulcorner\Gamma\urcorner \,|\, \varphi$ be given. If $\mathcal{C}$ is a generating set for $\mathcal{E}$, then*

$$\langle A, \alpha\rangle \Vdash_{[\![\Gamma]\!]} [\![\varphi]\!]$$

just in case $\langle A, \alpha\rangle \Vdash \varphi[a]$ for each $a \in_C A$, $C \in \mathcal{C}$.

PROOF. Let $\langle A, \alpha\rangle \Vdash [\![\varphi]\!]$, so for every $p{:}\langle A, \alpha\rangle \twoheadrightarrow [\![\Gamma]\!]$, we have $\langle A, \alpha\rangle \Vdash \varphi(p)$. Clearly, for every $a \in_C A$, $C \in \mathcal{C}$, then, $C \Vdash \varphi(pa)$, so $\langle A, \alpha\rangle \Vdash \varphi[a]$.

On the other hand, suppose that $\langle A, \alpha\rangle \Vdash \varphi[a]$ for all $a \in_C A$, $C \in \mathcal{C}$. Let $p{:}\langle A, \alpha\rangle \twoheadrightarrow [\![\Gamma]\!]$ be given. Then, for each $a \in_C A$, $C \in \mathcal{C}$, we have $C \Vdash \varphi(pa)$. Hence, $p \circ a$ equalizes $\mathsf{Coker}([\![\varphi]\!] \rightarrowtail [\![\ulcorner\Gamma\urcorner]\!])$ for each $a{:}C \twoheadrightarrow A$, $C \in \mathcal{C}$ and thus (by the assumption that $\mathcal{C}$ is a generating set for $\mathcal{E}$), p equalizes $\mathsf{Coker}([\![\varphi]\!] \rightarrowtail [\![\ulcorner\Gamma\urcorner]\!])$, too. Hence p factors through $[\![\varphi]\!]$. Since p was an arbitrary homomorphism, we see $\langle A, \alpha\rangle \Vdash_{[\![\Gamma]\!]} [\![\varphi]\!]$. □

It is natural to ask whether this semantics comes with a Kripke-Joyal style theorem, similar to Theorem 4.3.1. Unfortunately, we do not have any such theorem relating the condition that $\langle A, \alpha\rangle \Vdash \varphi[a]$ and the structure of φ.

The motivation for this section is the intuition that, in order to show that a coalgebra $\langle A, \alpha\rangle$ forces a coequation $[\![\varphi]\!]$, one checks that, for each element of $a \in A$ and homomorphism $p{:}\langle A, \alpha\rangle \twoheadrightarrow [\![\Gamma]\!]$, $p(a) \in [\![\varphi]\!]$. In other words, in practice, one may verify that $\langle A, \alpha\rangle$ forces $[\![\varphi]\!]$ at each $a \in A$.

Supposing that, in fact, $\langle A, \alpha\rangle$ does not force $[\![\varphi]\!]$, one may still be interested in those elements $a \in A$ that do force φ. In what remains, we will define an functor J_φ taking coalgebras $\langle A, \alpha\rangle$ to the subobject $B \leq A$ consisting of all those elements of A which pointwise force φ. We conclude by showing that if $[\![\varphi]\!]$ is a coequation over an injective $[\![\Gamma]\!]$ (i.e., a proper coequation, rather than a conditional coequation), then we can define the comonad $\mathbb{G}^\varphi$ in terms of J_φ and $[-]$.

Given $\ulcorner\Gamma\urcorner \,|\, \varphi$ and $\langle A, \alpha\rangle \in \mathcal{E}_{\mathbb{G}}$, we define

$$J_\varphi\langle A, \alpha\rangle = \bigvee\{S \leq A \mid \forall p{:}\langle A, \alpha\rangle \longrightarrow [\![\Gamma]\!] \,.\, \exists_p S \leq [\![\varphi]\!]\}.$$

It is easy to check that for all $p{:}\langle A, \alpha\rangle \twoheadrightarrow [\![\Gamma]\!]$,

$$\exists_p J_\varphi\langle A, \alpha\rangle \leq [\![\varphi]\!].$$

In other words, $\exists_p J_\varphi\langle A, \alpha\rangle \leq J_\varphi\langle A, \alpha\rangle$.

THEOREM 4.4.3. *$J_\varphi\langle A, \alpha\rangle$ is invariant, in the sense of Section 3.8. That is, for every homomorphism $p\colon\langle A, \alpha\rangle \twoheadrightarrow \langle A, \alpha\rangle$,*

$$\exists_p J_\varphi\langle A, \alpha\rangle \leq J_\varphi\langle A, \alpha\rangle.$$

PROOF. Let $p\colon\langle A, \alpha\rangle \twoheadrightarrow \langle A, \alpha\rangle$ be given. It suffices to show that, for every $r\colon\langle A, \alpha\rangle \twoheadrightarrow [\![\Gamma]\!]$,

$$\exists_r \exists_p J_\varphi\langle A, \alpha\rangle \leq [\![\varphi]\!],$$

i.e., $\exists_{r\circ p} J_\varphi\langle A, \alpha\rangle \leq [\![\varphi]\!]$. This follows, since $J_\varphi\langle A, \alpha\rangle$ is a join of subobjects S such that $\exists_{r\circ p} S \leq [\![\varphi]\!]$ and joins commute with $\exists$. $\square$

We still must show that J_φ defines a functor. Let $f\colon\langle B, \beta\rangle \twoheadrightarrow \langle A, \alpha\rangle$ be given. We will show that $\exists_f J_\varphi\langle B, \beta\rangle \leq J_\varphi\langle A, \alpha\rangle$. This allows one to define $J_\varphi f$ to be the composite along the top row of Figure 3.

$$\begin{tikzcd} J_\varphi\langle B, \beta\rangle \arrow[r, two heads] \arrow[d, tail] & \exists_f J_\varphi\langle B, \beta\rangle \arrow[r, dotted, tail] \arrow[dr, tail] & J_\varphi\langle A, \alpha\rangle \arrow[d, tail] \\ \langle B, \beta\rangle \arrow[rr, "f"'] & & \langle A, \alpha\rangle \end{tikzcd}$$

FIGURE 3. Definition of J_φ on arrows.

To show that $\exists_f J_\varphi\langle B, \beta\rangle \leq J_\varphi\langle A, \alpha\rangle$, it suffices to show that, for every homomorphism $p\colon\langle A, \alpha\rangle \twoheadrightarrow [\![\Gamma]\!]$,

$$\exists_p \exists_f J_\varphi\langle B, \beta\rangle = \exists_{p\circ f} J_\varphi\langle B, \beta\rangle \leq [\![\varphi]\!].$$

But this is clear, since $p \circ f$ is a homomorphism.

THEOREM 4.4.4. *Let $a \in_B A$. Then $a \in J_\varphi\langle A, \alpha\rangle$ just in case $\langle A, \alpha\rangle \Vdash \varphi[a]$.*

PROOF. Clearly, if $\langle A, \alpha\rangle \Vdash \varphi[a]$, then $a \in J_\varphi A$.

On the other hand, suppose that $a \in J_\varphi A$ and let $r\colon\langle A, \alpha\rangle \twoheadrightarrow [\![\Gamma]\!]$ be given. Then

$$\exists_r J_\varphi A = \bigvee\{\exists_r S \leq A \mid \forall p\colon\langle A, \alpha\rangle \longrightarrow \ulcorner\Gamma\urcorner \,.\, \exists_p S \leq [\![\varphi]\!]\} \leq [\![\varphi]\!].$$

Hence, $ra \in [\![\varphi]\!]$. $\square$

From Chapter 3, we know that there is a comonad

$$\mathbb{G}^\varphi\colon \mathcal{E}_{\mathbb{G}} \longrightarrow \mathcal{E}_{\mathbb{G}},$$

$\mathbb{G}^\varphi = \langle G^\varphi, \varepsilon^\varphi, \delta^\varphi\rangle$, such that $\langle A, \alpha\rangle \Vdash [\![\varphi]\!]$ just in case $\langle A, \alpha\rangle \perp \varepsilon^\varphi_\alpha$. In fact, $G^\varphi\langle A, \alpha\rangle$ is the greatest subcoalgebra $\langle B, \beta\rangle$ of $\langle A, \alpha\rangle$ such that $\langle B, \beta\rangle \Vdash [\![\varphi]\!]$. Hence, there is some similarity between $G^\varphi\langle A, \alpha\rangle$ and our definition of $J_\varphi\langle A, \alpha\rangle$. The following theorem makes the relationship between the two clearer.

THEOREM 4.4.5. *If* $[\![\Gamma]\!]$ *is injective (so that* $[\![\varphi]\!]$ *defines a covariety, rather than a quasi-covariety), then*

$$G^{\langle}A,\, \alpha\rangle = [J_\varphi\langle A,\, \alpha\rangle]_\alpha.$$

In other words, $G^\varphi\langle A,\, \alpha\rangle$ *is the largest subcoalgebra of* $J_\varphi\langle A,\, \alpha\rangle$*.*

PROOF. Since $G^{\langle}A,\, \alpha\rangle \Vdash [\![\varphi]\!]$, it follows that for every $p\colon\langle A,\, \alpha\rangle \to [\![\Gamma]\!]$, we have $\exists_p UG^\varphi\langle A,\, \alpha\rangle \le [\![\varphi]\!]$. Hence, by definition of J_φ, $UG^\varphi\langle A,\, \alpha\rangle \le J_\varphi\langle A,\, \alpha\rangle$ and so $G^\varphi\langle A,\, \alpha\rangle \le [J_\varphi\langle A,\, \alpha\rangle]$.

On the other hand, to prove the reverse inclusion, it suffices to show that the coalgebra $[J_\varphi\langle A,\, \alpha\rangle]$ forces the coequation $[\![\varphi]\!]$. That is, for every homomorphism

$$p\colon [J_\varphi\langle A,\, \alpha\rangle] \longrightarrow [\![\Gamma]\!],$$

$\mathsf{Im}\, p \le [\![\varphi]\!]$. Since $[\![\Gamma]\!]$ is injective, p extends to a homomorphism

$$\widetilde{p}\colon \langle A,\, \alpha\rangle \longrightarrow [\![\Gamma]\!].$$

Since $\exists_{\widetilde{p}} J_\varphi\langle A,\, \alpha\rangle \le [\![\varphi]\!]$, the conclusion follows. $\square$

Concluding remarks and further research

In this thesis, we had three main goals in mind. First, we wanted to develop the theory of coalgebras alongside the theory of algebras in a general setting. Second, we wanted to apply the principle of duality to some well-known and fundamental theorems of universal algebra to learn their implications in the theory of coalgebras. Lastly we wanted to provide an internal logic for categories of coalgebras which is appropriate for representing relevant constructions and for expressing the relation between $\mathcal{E}$ and $\mathcal{E}_{\mathbb{G}}$ via certain transfer principles.

The first task yielded sufficient conditions for a category of coalgebras to be well enough behaved for the development of basic results like the co-Birkhoff theorem. Among other results, we found that a category $\mathcal{E}_{\mathbb{G}}$ of coalgebras for a comonad $\mathbb{G}$ inherits much of the relevant structure from $\mathcal{E}$ presuming $\mathcal{E}$ has epi-regular mono factorizations and cokernel pairs and $\mathbb{G}$ preserves regular monos. If we further assume that $\mathcal{E}$ has enough injectives, then so does $\mathcal{E}_{\mathbb{G}}$, and these injectives provide a natural interpretation of coequations. We also showed that $\mathcal{E}_{\mathbb{G}}$ is "as complete" as $\mathcal{E}$ is, although the limits in $\mathcal{E}_{\mathbb{G}}$ are not created by the forgetful functor. Supposing that $\mathcal{E}$ is a locally complete logos with regular epi-regular mono factorizations, and $\mathbb{G}$ nearly preserves pullbacks, then $\mathcal{E}_{\mathbb{G}}$ is also a locally complete logos and thus interprets first order logic.

We further contributed to the theory of coalgebras by offering a new definition of bisimulation which is, we hope, more natural in settings in which choice is not available. This definition preserves the intuition behind bisimulation — two elements are bisimilar just in case there's a coalgebraic relation $\langle R, \rho\rangle$ such that they are related by the image of R. Furthermore, while it allows for greater structure than the traditional definition in categories without choice, it also reduces to that definition if choice is available (or if $\mathbb{G}$ preserves pullbacks).

The second task is closely related to the first. In order to dualize familiar theorems from universal algebra, one must first state and prove these theorems in categorical terms. In this stage, one sees what is really relevant, categorically speaking, for a theorem like, say, the variety theorem and this in turn helps guide the development of the theory of coalgebras. To the extent that we are interested in the duals of such fundamental algebraic theorems, we are committed in assuming the dual conditions (with certain exceptions – the proof of the subdirect product theorem is an example of a proof which is not easily dualized. Our approach, following [**GS98**], involved finding an alternate proof.).

Once a classic theorem has been stated and proved in terms which are easily dualized, the dual theorem must still be interpreted. For the variety theorem, this meant understanding coequations as predicates over the carrier of a cofree coalgebra, and coequation forcing as an assertion about the images of a coalgebra under the various colorings. This in turn led to an understanding of the invariance theorem. Namely, it allowed a definition of a modal operator which takes a coequation to the largest subobject which is invariant under all colorings. Without interpretations such as these, the result of dualizing a theorem is largely formal — we receive a provable statement but are at a certain loss for what it *means*.

The final task, too, relied on the first task for establishing the inheritance in $\mathcal{E}_{\mathbb{G}}$ of the relevant structure in $\mathcal{E}$. This established that an internal logic for $\mathcal{E}_{\mathbb{G}}$ could include full first-order logic. The work on bisimulations suggested a closure operator for the language $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$, in which the closed propositions correspond to n-simulations. The work on the invariance theorem suggested an interior operator as well, taking each proposition (i.e., conditional coequation) to its largest invariant subcoalgebra. For each of these operators, however, there were important properties which are semantically verifiable but not expressible in the internal logic – unless $\mathcal{E}_{\mathbb{G}}$ has exponentials.

The relation between $\mathcal{E}$ and $\mathcal{E}_{\mathbb{G}}$ suggested the addition of certain transfer rules which allow one to make inferences in $\mathcal{L}(\mathcal{E}_{\mathbb{G}})$ based on derivations in $\mathcal{L}(\mathcal{E})$ and vice versa. These transfer rules allowed the characteristic property of cofree coalgebras to be expressed in a natural way in the join of the logics involved.

The work presented here can be extended in several ways. First, one may be interested in base categories with less structure than we've assumed. For instance, if one considers coalgebras over various categories of posets, then the assumption of "enough injectives" is unreasonable. Hence, it would be worthwhile to investigate what structural properties may be lost in such settings and to try to understand what the appropriate notion of a coequation is in these settings.

Related to this concern is a question that has, unfortunately, largely remained unanswered in this thesis. Namely, what applications are there for coalgebras over categories other than **Set** and related categories? There is a notable lack of examples of such coalgebras – although, one should stress that the broad approach developed here does not depend on mathematical applications for its justification. Rather, it is motivated by a desire to make clear which theorems of universal algebra can be dualized in a straightforward way. Since we are not interested in coalgebras over $\mathbf{Set}^{\mathsf{op}}$, this means that we must make clear what properties of **Set** are relevant in the classical theorem, so that we can see whether these properties are reasonable for categories of coalgebras as well. Nonetheless, compelling examples of coalgebras over other categories would be most helpful in understanding the basic theory.

The project of dualizing theorems in universal algebra is still in its infancy. One can go through any standard text in universal algebra and find candidates for dualization. One needs, however, to develop a few methods for dualizing certain common assertions in algebra. A survey of the work of Andréka and Németi [**AN83, Ném82, AN81a, AN81b, AN79b, AN79a, AN78**] shows great promise in this direction. In the early 1980's, they extended the work of Herrlich and Banaschewski [**BH76**] to give an analysis of "cone-injectivity" and classes of algebras defined by an extension of equational logic. This work was unknown to the author until an anonymous reviewer for CMCS 2001 brought it to his attention. A review of these earlier results, with an eye towards applications of their coalgebraic dual, seems most promising.

The development of the internal logic in Chapter 4 should lead to clean proofs of certain claims about coalgebras. One would certainly like more examples of such proofs. To begin, it is reasonable to take well-known properties of (certain) coalgebras and prove them in the internal logic, as we did with the property of coinduction for the final coalgebra and also the proof that bisimulations compose (given that $\mathbb{G}$ preserves regular relations). However, time did not permit as broad a development of these proofs in the internal logic as one would like, and in particular, we did not attempt to represent the property of corecursion and prove that it holds in the final coalgebra. The aim of using the internal logic to re-prove well-known results is two-fold: First, it gives a measure of the practical strength of the logic and shows how the transfer principles can be used, and second, it allows one to develop skills of reasoning internally, much simplifying (and formalizing) proofs, and this skill can be applied for "real" advances to the theory as well.

APPENDIX A

Preliminaries

A.1. Notation

We adopt the following notation conventions for morphisms in a category.

Morphism	Arrow
Monos	$i : A \rightarrowtail B$
Epis	$p : A \twoheadrightarrow B$
Regular monos	$i : A \rightarrowtail B$
Regular epis	$p : A \twoheadrightarrow B$
Isomorphisms	$i : A \rightarrowtail B$
Natural transformations	$\tau : F \Rightarrow G$
Cones	$\tau : A \Rightarrow G$

Table 1. Notation conventions

A.2. Factorization systems

This section gives a brief review of factorization systems with a special emphasis on the factorization systems of special interest here: regular epi-mono factorizations (for categories of algebras) and epi-regular mono factorizations (for categories of coalgebras). For a more thorough treatment of factorization systems, see [**Bor94**, Volume I] or [**AHS90**]. First, we review the definition of regular epi/regular mono.

Definition A.2.1. We say that a map $p : A \twoheadrightarrow B$ is a *regular epi* if there is a pair of maps e_1 and e_2 such that

$$\bullet \underset{e_2}{\overset{e_1}{\rightrightarrows}} A \xrightarrow{p} B$$

is a coequalizer diagram. Dually, a *regular mono* is a map that is an equalizer of some pair of arrows.

Throughout, we will often use the fact that regular epis are strong, so we include a definition and proof of this connection.

Definition A.2.2. An epi e is *strong* just in case, whenever the square below commutes, with m mono, then there is a (necessarily unique) map d making each triangle commute.

$$\begin{tikzcd}
\bullet \arrow[r, two heads, "e"] \arrow[d] & \bullet \arrow[d] \arrow[dl, dotted, "d"'] \\
\bullet \arrow[r, tail, "m"'] & \bullet
\end{tikzcd}$$

A *strong mono* is a mono m as in the diagram above such that, whenever the square commutes and e epi, then again there is a unique d making the triangles commute.

THEOREM A.2.3. *Every regular epi is strong (and, dually, every regular mono is strong).*

PROOF.

$$\begin{tikzcd}
\bullet \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] & \bullet \arrow[r, "e"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \arrow[dl, dotted, "d"'] \\
& \bullet \arrow[r, tail, "m"'] & \bullet
\end{tikzcd}$$

FIGURE 1. Every regular epi is strong.

Let e be the coequalizer of k_1, k_2 as shown in Figure 1 and let m be a mono making the diagram commute. Then it is easy to see that (because m is monic), f also coequalizes k_1 and k_2 and so there is a unique d making the upper triangle commute. The lower triangle also commutes, since e is epi. □

DEFINITION A.2.4. A factorization system for a category $\mathcal{C}$ consists of a pair $\langle\mathcal{E}, \mathcal{M}\rangle$ where $\mathcal{E}$ and $\mathcal{M}$ are class of morphisms of $\mathcal{C}$ satisfying the following:

(1) Every isomorphism is in $\mathcal{E}$ and $\mathcal{M}$;
(2) $\mathcal{E}$ and $\mathcal{M}$ are closed under composition;
(3) Whenever $e \in E$ and $m \in M$ such that the square below commutes, there is a unique d as shown, making each triangle commute.

$$\begin{tikzcd}
\bullet \arrow[r, "e"] \arrow[d] & \bullet \arrow[d] \arrow[dl, dotted, "d"'] \\
\bullet \arrow[r, "m"'] & \bullet
\end{tikzcd}$$

(4) For each $f{:}A{\to}B$ in $\mathcal{C}$, there is an $e{:}A{\to}C$ in $\mathcal{E}$ and a $m{:}C{\to}B$ in $\mathcal{M}$ such that $f = m \circ e$ (as shown below).

$$\begin{tikzcd}
A \arrow[rr, bend left, "f"] \arrow[dr, "e"'] & & B \\
& C \arrow[ur, tail, "m"'] &
\end{tikzcd}$$

Note that our definition of factorization system does not require that $\mathcal{E}$ be a subclass of the epis of $\mathcal{C}$ or that $\mathcal{M}$ be a subclass of the monos. Nonetheless, the common examples of factorization systems do have this property, and certainly the factorization systems in which we are interested are no exception.

THEOREM A.2.5. *Let $\langle\mathcal{E},\,\mathcal{M}\rangle$ be a factorization system for $\mathcal{C}$. Factorizations $f = m \circ e$, where $e \in \mathcal{E}$ and $m \in \mathcal{M}$, are unique up to isomorphism.*

PROOF. If $m \circ e = m' \circ e'$, where $e,\ e' \in \mathcal{E}$ and $m,\ m' \in \mathcal{M}$, then there are unique $d,\ d'$, as shown in Figure 2, making the triangles commute. The uniqueness part of Condition 3 from Definition A.2.4 implies that the composites $d \circ d'$ and $d' \circ d$ are the identity. □

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "e'"'] & C \arrow[d, "m"] \arrow[dl, dotted, bend left=10, "d"] \\
D \arrow[r, "m'"'] \arrow[ur, dotted, bend left=10, "d'"] & C
\end{tikzcd}$$

FIGURE 2. $\langle\mathcal{E},\,\mathcal{M}\rangle$-factorizations are essentially unique.

For the remainder of this section, let $\mathcal{E}_{\mathrm{e}}$ denote the epis of $\mathcal{C}$ and $\mathcal{M}_{\mathrm{m}}$ the monos. Also, let $\mathcal{E}_{\mathrm{re}}$ denote the regular epis and $\mathcal{M}_{\mathrm{rm}}$ the regular monos. We complete our review of factorization systems by introducing the notion of regular epi-mono factorizations and its dual, epi-regular mono factorizations. We show that, if every map factors by a regular epi followed by a mono, then $\langle\mathcal{E}_{\mathrm{re}},\,\mathcal{M}_{\mathrm{m}}\rangle$ is a factorization system (and the dual result as well).

DEFINITION A.2.6. Let $\mathcal{E}$ be a category. We say that $\mathcal{E}$ *has regular epi-mono* factorizations if every arrow $f\colon A \to B$ can be factored into a regular epi followed by a mono.

$$\begin{tikzcd}
A \arrow[rr, bend left=10, "f"] \arrow[dr] & & B \\
& A/f \arrow[ur] &
\end{tikzcd}$$

The codomain of the regular epi is denoted A/f, as shown above.

Dually, we say that $\mathcal{E}$ *has epi-regular mono* factorizations if every arrow $f\colon A \to B$ can be factored into a epi followed by a regular mono.

$$\begin{tikzcd}
A \arrow[rr, bend left=10, "f"] \arrow[dr] & & B \\
& \mathsf{Im}(f) \arrow[ur] &
\end{tikzcd}$$

The domain of the regular mono is denoted $\mathsf{Im}(f)$, as shown above.

THEOREM A.2.7. *If $\mathcal{E}$ has regular epi-mono factorizations (epi-regular mono factorizations, resp.), then $\langle \mathcal{E}_{re}, \mathcal{M}_m \rangle$ ($\langle \mathcal{E}_e, \mathcal{M}_{rm} \rangle$, resp.) form a factorization system.*

PROOF. Conditions (1) and (2) are obvious, and (4) is by hypothesis. The diagonal condition (by (3)) is just the fact that regular epis (monos, resp.) are strong. $\square$

As we can see in the proof of Theorem A.2.7, the strong epis provide most of the properties we require. Indeed, throughout this thesis, the assumption of regular epi-mono factorizations in $\mathcal{E}^\Gamma$ could be largely replaced by strong epi-mono factorizations, weakening some assumptions while strengthening others in the process. We nonetheless prefer to stick with the regular epis, since in the algebraic setting, they correspond to deductively closed sets of equations. We also use epi-regular mono factorizations in $\mathcal{E}_\Gamma$ in keeping with the duality.

We close this section with a categorical definition of the axiom of choice.

DEFINITION A.2.8. We say that an epi p (a mono i, resp.) *splits* if there is a map f such that $p \circ f = \mathsf{id}$ ($f \circ i = \mathsf{id}$, resp.) Such epis (monos, resp.) are necessarily regular.

DEFINITION A.2.9. Let $\mathcal{E}$ be given. We say that $\mathcal{E}$ *satisfies the axiom of choice* if every epi splits. That is, if for every epi p, there is a (necessarily monic) i such that

$$p \circ i = \mathsf{id}$$

.

We say that $\mathcal{E}$ *satisfies the weak axiom of choice* if every regular epi splits.

THEOREM A.2.10. *If $\mathcal{E}$ satisfies the weak axiom of choice, then every endofunctor $\Gamma : \mathcal{E} \to \mathcal{E}$ preserves regular epis.*

PROOF. Let p be a regular epi in $\mathcal{E}$. Then p splits, and hence Γp splits. $\square$

A.3. Predicates and Subobjects

We very briefly present the basic construction of the category $\mathsf{Sub}(A)$ and show how to define $\wedge$ and $\vee$ in $\mathsf{Sub}(A)$. This material is not intended to be complete. In particular, we simply show the constructions here without bothering to verify that our construction of $\wedge$ (say) really does define a meet operation. For a proper introduction in lattice theory, see [**DP90**], and for a discussion of the Heyting algebra $\mathsf{Sub}(A)$ in a topos $\mathcal{E}$, see [**LM92**].

Let $\mathcal{C}$ be a category and $A \in \mathcal{C}$. We form the category, $\mathsf{Sub}_{\mathcal{C}}(A)$ or just $\mathsf{Sub}(A)$, as follows: Take the full subcategory of the slice category $\mathcal{C}/A$ consisting of the monos

$$P \rightarrowtail A \,.$$

Then, take the quotient of that subcategory by the relation $\cong$ that holds if two objects are isomorphic. In other words, we consider the skeleton of the category of monos into A. We call the elements of $\mathsf{Sub}(A)$ the *subobjects* of A. It is easy to see that $\mathsf{Sub}(A)$ is a poset.

We define the intersection of two subobjects P and Q as the pullback,

$$\begin{tikzcd} P \wedge Q \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, tail] \\ Q \arrow[r, tail] & A \end{tikzcd}$$

if it exists. More generally, the intersection $\bigwedge P_i$ of a collection of subobjects P_i of A is the generalized pullback of the P_i's.

In a category with $+$ and a factorization system $\langle \mathcal{E}, \mathcal{M} \rangle$, $\mathcal{E}$ a subclass of the epis and $\mathcal{M}$ a subclass of the monos, the *join* (or *union*) of two subobjects as the factorization of the induced map $P + Q \longrightarrow A$:

$$\begin{tikzcd} P + Q \arrow[r, "\triangleright" description] \arrow[rr, bend left] & P \vee Q \arrow[r, tail] & A \end{tikzcd} .$$

More generally, in a category with arbitrary coproducts, one can define an infinite join, $\bigvee_i P_i$ of subobjects P_i.

Definition A.3.1. A category $\mathcal{C}$ is *well-powered* if each object has set-many subobjects. $\mathcal{C}$ is *regularly well-powered* if each object has set-many regular subobjects.

Dually, $\mathcal{C}$ is (regularly, resp.) *co-well-powered* if, for each object C, there are set-many (regular, resp.) epis out of C, up to isomorphism.

In a well-powered category $\mathcal{C}$ with pullbacks, we have a contravariant functor

$$\mathsf{Sub} : \mathcal{C}^{\mathsf{op}} \longrightarrow \mathbf{Poset} .$$

We must describe the action of Sub on arrows $f : A \to B$, which we write as

$$f^* : \mathsf{Sub}(B) \longrightarrow \mathsf{Sub}(A) .$$

Take a subobject $P \rightarrowtail B$ to the object making this square a pullback:

$$\begin{tikzcd} f^*(P) \arrow[r] \arrow[d, tail] & P \arrow[d, tail] \\ A \arrow[r] & B \end{tikzcd}$$

If $\mathcal{C}$ has regular epi-mono factorizations, then f^* has a left adjoint, denoted $\exists_f$ and defined by taking the factorization shown below.

$$\begin{tikzcd} P \arrow[r, "\triangleright" description] \arrow[d, tail] & \exists_f P \arrow[d, tail] \\ A \arrow[r] & B \end{tikzcd}$$

See Section 4.1 for a discussion of a right adjoint to f^*.

We complete our brief review of subobjects by showing that a regularly well-powered category is also regularly co-well-powered, given kernel pairs.

CLAIM A.3.2. *If $\mathcal{C}$ has kernel pairs and is regularly well-powered, then $\mathcal{C}$ is regularly co-well-powered. Dually, a regularly co-well-powered category with cokernel pairs is regularly well-powered.*

PROOF. Assume $\mathcal{C}$ is finitely complete and regularly well-powered and $C \in \mathcal{C}$. Then, we map quotients of C to regular subobjects of $C \times C$ by taking a regular epimorphism q to its kernel pair. This mapping is injective. $\square$

A.3.1. Regular subobjects. The categories of coalgebras in which we are interested do not, in general, have regular epi-mono factorizations. Rather, they have epi-regular mono factorizations. Consequently, the corresponding category of subobjects is not well-behaved: we cannot define the join of arbitrary subcoalgebras.

If $\mathcal{C}$ is a category with epi-regular mono factorizations, it is natural to consider the regular subobjects of A as predicates over A. In the category $\mathsf{RegSub}_{\mathcal{C}}(A)$ of regular subobjects, one can define meet, join, etc., as before and view the collection of regular subobjects as the predicates over A.

A.4. Relations

We briefly introduce the basic definitions for relations on a category. Since we are concerned with categories with finite products, for the most part, we simplify this material by assuming finite products exist whenever convenient. See [**Bor94**, Volume 2, Chapter 2] for a more complete discussion of this topic.

DEFINITION A.4.1. A collection of maps $\{f_i : A \to B_i\}_i \in I$ are *jointly monic* if, whenever $g, h : C \to A$ satisfy, for all $i \in I$,

$$f_i \circ g = f_i \circ h$$

then $g = h$.

If $I = 1$, then jointly monic is just monic.

DEFINITION A.4.2. Let $\mathcal{C}$ be a category, A and B objects of $\mathcal{C}$. A *(binary) relation on A and B* is a triple $\langle R, r_1, r_2\rangle$ such that

$$r_1 : R \longrightarrow A,$$
$$r_2 : R \longrightarrow B$$

and r_1 and r_2 are jointly monic. This definition generalizes in the obvious way to n-ary (or I-ary) relations. A unary relation is a subobject.

If $A = B$, we say that $\langle R, r_1, r_2\rangle$ is a *relation on A*. Also, we often refer to a relation $\langle R, r_1, r_2\rangle$ by just its carrier R, if no confusion will result.

If $\mathcal{C}$ has finite products, then a relation $\langle R, r_1, r_2\rangle$ on A and B is just a subobject $\langle R, \langle r_1, r_2\rangle\rangle$ of $A \times B$. Also, any pullback (and so, any kernel pair) is a relation. In particular, $\Delta_A = \langle A, \mathsf{id}_A, \mathsf{id}_A\rangle$ is a relation on A (sometimes called the *equality* relation or the *diagonal*) and, more generally, given a map $f : A \to B$, then $\langle A, \mathsf{id}_A, f\rangle$ is a relation on A and B, called the *graph* of f (denoted $\mathsf{graph}(f)$).

The category of relations on A and B forms a partial order, where $\langle R, r_1, r_2\rangle \leq \langle S, s_1, s_2\rangle$ just in case there is an arrow $f : R \to S$ such that

$$r_1 = s_1 \circ f, r_2 \qquad\qquad = s_2 \circ f.$$

Given finite products, this ordering is just the same as the ordering on $\mathsf{Sub}(A \times B)$, of course.

In a category $\mathcal{C}$ with finite products and epi-regular mono factorizations, we can define the composition of two relations easily. Namely, let $\langle R, r_1, r_2\rangle$ be a relation on A and B and $\langle S, s_1, s_2\rangle$ a relation on B and C. Take the pullback shown in Figure 3. In general, this will not be a relation, so take the regular epi-mono factorization of $P \to A \times C$.

$$\begin{tikzcd}
 & & \bullet \arrow[dl, "t_1"'] \arrow[dr, "t_2"] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
 & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & {} & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\
A & & B & & C
\end{tikzcd}$$

FIGURE 3. Composition of relations

If R is any relation on A, we say that R is *reflexive* if $\Delta \leq R$.

Given any relation $\langle R, r_1, r_2\rangle$, the triple $\langle R, r_2, r_1\rangle$ is also a relation, called the *opposite relation of* R and denoted R^0. We say that R is *symmetric* if $R^0 \leq R$.

Because $-^0$ is monotone and $(R^0)^0 = R$, we have that a relation R is symmetric iff $R^0 = R$.

A relation R on A is said to be *transitive* if $R \circ R \leq R$. If R is reflexive, then $R \leq R \circ R$. Thus, if R is reflexive, then R is transitive iff $R \circ R = R$.

Definition A.4.3. A relation R on A is an *equivalence relation* if it is reflexive, symmetric and transitive.

Notice that a kernel pair of an arrow is always an equivalence relation. We say that an equivalence relation is *effective* if it is the kernel pair of its coequalizer.

$$
\begin{tikzcd}
\Delta_A \arrow[rr, bend left, tail] \arrow[dr, shift left, "\mathsf{id}"] \arrow[dr, shift right, "\mathsf{id}"'] & & R \arrow[dl, shift right, "r_2"'] \arrow[dl, shift left, "r_1"] \\
& A &
\end{tikzcd}
\quad
\begin{tikzcd}
R^0 \arrow[rr, bend left, tail] \arrow[dr, shift left, "r_1"] \arrow[dr, shift right, "r_2"'] & & R \arrow[dl, shift right, "r_2"'] \arrow[dl, shift left, "r_1"] \\
& A &
\end{tikzcd}
\quad
\begin{tikzcd}
R \circ R \arrow[rr, bend left, tail] \arrow[dr, shift left, "r_2 \circ t_2"] \arrow[dr, shift right, "r_1 \circ t_1"'] & & R \arrow[dl, shift right, "r_2"'] \arrow[dl, shift left, "r_1"] \\
& A &
\end{tikzcd}
$$

Figure 4. The defining conditions for equivalence relations.

The equality relation Δ is an effective equivalence relation, and is obviously the least equivalence relation. Also, in **Set**, for instance, every equivalence relation is effective.

Definition A.4.4. A category $\mathcal{C}$ is *regular* if it satisfies the following:

- Every arrow has a kernel pair.
- Every kernel pair has a coequalizer.
- The pullback of a regular epi is a regular epi (regular epis are *stable under pullbacks*.

A regular category in which all equivalence relations are effective is called *exact.*

A.5. Monads and comonads

This section is a brief reminder of the basic definition of monad and how a pair of adjoint functors give rise to a monad. See any basic text on category theory for more details. We take this material largely from [**BW85, Bor94**].

Definition A.5.1. A *monad* (also called a *triple*) is an ordered triple $\mathbb{T} = \langle T, \eta, \mu \rangle$ where

$$T : \mathcal{C} \longrightarrow \mathcal{C}$$

is an endofunctor,

$$\eta : 1_{\mathcal{C}} \Longrightarrow T \text{ and}$$
$$\mu : T^2 \Longrightarrow T$$

are natural transformations such that the following diagrams commute.

$$\begin{tikzcd} T^3 \arrow[r, Rightarrow, "\mu_T"] \arrow[d, Rightarrow, "T\mu"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\ T^2 \arrow[r, Rightarrow, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, Rightarrow, "\eta_T"] \arrow[dr, equal] & T^2 \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "T\eta"'] \arrow[dl, equal] \\ & T & \end{tikzcd}$$

The first diagram is called the *associativity condition* and the second the *unit condition.*

Rather than give explicit examples of monads, let us show how any adjoint pair gives rise to a monad. In Section 2.1, we state the Eilenberg-Moore theorem showing that every monad arises from an adjoint pair. In fact, it arises from (at least) two different pairs of adjoints, but we will not discuss the Kleisli construction. See any of [**Bor94, Lan71, BW85**] for a more thorough development of this topic.

Let $L:\mathcal{C}\to\mathcal{D}$ and $R:\mathcal{D}\to\mathcal{C}$ be given, with $L\dashv R$. Let $\eta:\mathsf{id}_{\mathcal{C}}\Rightarrow RL$ and $\varepsilon:LR\Rightarrow\mathsf{id}_{\mathcal{D}}$ be the unit and counit of the adjunction, respectively. It is easy to show that

$$\langle RL,\, \eta,\, R\varepsilon_L\rangle$$

is a monad on $\mathcal{C}$. The associativity condition

$$\begin{tikzcd} RLRLRL \arrow[r, Rightarrow, "R\varepsilon_{LRL}"] \arrow[d, Rightarrow, "RLR\varepsilon_L"'] & RLRL \arrow[d, Rightarrow, "R\varepsilon_L"] \\ RLRL \arrow[r, Rightarrow, "R\varepsilon_L"'] & RL \end{tikzcd}$$

holds just by the naturality of ε. The unit condition

$$\begin{tikzcd} RL \arrow[r, Rightarrow, "\eta_{RL}"] \arrow[dr, equal] & RLRL \arrow[d, Rightarrow, "R\varepsilon_L"] & RL \arrow[l, Rightarrow, "RL\eta"'] \arrow[dl, equal] \\ & RL & \end{tikzcd}$$

holds just because of the identities

$$\eta_R \circ R\varepsilon = \mathsf{id}_{\mathcal{C}} \ \text{ and}$$
$$L\eta \circ \varepsilon_L = \mathsf{id}_{\mathcal{D}}\,.$$

A comonad in $\mathcal{C}$ is a monad in $\mathcal{C}^{\mathsf{op}}$. We state the definition explicitly, nonetheless, since comonads play such an important role for categories of coalgebras.

Definition A.5.2. A *comonad* (also called a *cotriple*) is a triple $\mathbb{G}=\langle G,\,\varepsilon,\,\delta\rangle$ where $G:\mathcal{C}\to\mathcal{C}$ is a functor and

$$\varepsilon:G\Longrightarrow \mathsf{id}_{\mathcal{C}} \ \text{ and}$$
$$\delta:G\Longrightarrow G^2$$

are natural transformations such that the following diagrams commute.

$$\begin{tikzcd} G^3 & G^2 \arrow[l, Leftarrow, "\delta_G"'] \\ G^2 \arrow[u, Rightarrow, "G\delta"] & G \arrow[l, Leftarrow, "\delta"] \arrow[u, Rightarrow, "\delta"'] \end{tikzcd} \qquad \begin{tikzcd} G & G^2 \arrow[l, Leftarrow, "\varepsilon_G"'] \arrow[r, Rightarrow, "G\varepsilon"] & G^2 \\ & G \arrow[ul, equal] \arrow[u, Rightarrow, "\delta"'] \arrow[ur, equal] & \end{tikzcd}$$

One sees, by duality, that an adjoint pair also gives rise to a comonad. Explicitly, let $L \dashv R$, with unit η and counit ε. Then one easily shows that

$$\langle LR,\, \varepsilon,\, L\eta R\rangle$$

is a comonad.

Example A.5.3. Consider the adjoint pair

$$U\colon \mathbf{Grp} \longrightarrow \mathbf{Set} \text{ and}$$
$$F\colon \mathbf{Set} \longrightarrow \mathbf{Grp},$$

where U takes a group to its underlying set and F takes a set to the free group on that set. We have that $F \dashv U$. This yields a familiar monad on $\mathbf{Set}$, $\langle UF,\, \eta,\, \mu\rangle$. The unit of the monad,

$$\eta\colon \mathsf{id}_{\mathbf{Set}} \Longrightarrow UF,$$

is the insertion of generators $X \longrightarrow UFX$. The multiplication is a natural transformation

$$\mu\colon UFUF \Longrightarrow UF.$$

It can be described componentwise as follows: Given a set X, UFX is the set of group terms over X, which we can regard as finite strings over X. The set $UFUFX$, then, is the collection of group terms taking elements of UFX as variables. Thus, $UFUFX$ is the collection of finite strings over the "alphabet" UFX. The multiplication μ_X takes such a string and concatenates its elements, yielding a string over X.

The comonad $\langle FU,\, \varepsilon,\, \delta\rangle$ over $\mathbf{Grp}$ can be easily described too, although it may seem less familiar. The functor part of the comonad takes a group G to the free group over UG. The counit

$$\varepsilon_G\colon FUG \longrightarrow G$$

takes a term over G and multiplies it using the multiplication of G. The comultiplication

$$\delta_G\colon FUG \longrightarrow FUFUG$$

is given by $F\eta_{UG}$, where η is the insertion of generators described above. Thus, it is the group homomorphism extending this insertion to all of FUG.

Bibliography

[Acz88] P. Aczel, *Non-well-founded sets*, CSLI Lecture Notes, vol. 14, Stanford University, 1988.

[Adá01] Jiří Adámek, *Final coalgebras are ideal completions of initial algebras*, Journal of Logic and Computation (2001), To appear.

[AGM92] S. Abramsky, Dov M. Gabbay, and T. S. E Maibaum (eds.), *Handbook of logic in computer science*, vol. 1, Oxford Science Publications, 1992.

[AH00] Steve Awodey and Jesse Hughes, *The coalgebraic dual of Birkhoff's variety theorem*, Tech. Report CMU-PHIL-109, Carnegie Mellon University, Pittsburgh, PA, 15213, November 2000.

[AHS90] J. Adámek, H. Herrlich, and G. E. Strecker, *Abstract and concrete categories*, Wiley Interscience, 1990.

[AM89] Peter Aczel and Nax Mendler, *A final coalgebra theorem*, Category Theory and Computer Science, Lecture Notes in Computer Science, Vol. 389 (D. H. Pitt, D. E. Rydeheard, P. Dybjer, A. M. Pitts, and A. Poigne, eds.), Lecture Notes in Computer Science, vol. 389, Springer-Verlag, 1989.

[AN78] H. Andréka and I. Németi, *Loś lemma holds in every category*, Studia Sci. Math. Hungar. **13** (1978), 361–376.

[AN79a] H. Andréka and I. Németi, *Formulas and ultraproducts in categories*, Beitrage zur Algebra und Geometrie **8** (1979), 133–151.

[AN79b] H. Andréka and I. Németi, *Injectivity in categories to represent all first order formulas*, Demonstratio Mathematica **12** (1979), 717–732.

[AN81a] H. Andréka and I. Németi, *Cone-implicational subcategories and some birkhoff-type theorems*, Universal Algebra (Proc. Coll. Esztergom 1977), Colloq. Math. Soc. J. Bolyai, vol. 29, 1981, pp. 535–578.

[AN81b] H. Andréka and I. Németi, *A general axiomatizability theorem formulated in terms of cone-injective subcategories*, Universal Algebra (Proc. Coll. Esztergom 1977), Colloq. Math. Soc. J. Bolyai, vol. 29, North–Holland, Amsterdam, 1981, pp. 13–35.

[AN83] H. Andréka and I. Németi, *Generalization of variety and quasivariety concept to partial algebras through category theory*, Dissertationes Mathematicae (Rozprawy Math.) (1983).

[AP01] Jiří Adámek and Hans-E. Porst, *From varieties of algebras to covarieties of coalgebras*, In Corradini et al. [**CLM01**].

[AR94] Jiří Adámek and Jiří Rosický, *Locally presentable and accessible categories*, London Mathematical Society Lecture Note Series, vol. 189, Cambridge University Press, 1994.

[Bar92] Michael Barr, *Algebraically compact functors*, Journal for Pure and Applied Algebra **82** (1992), 211–231.

[Bar93] M. Barr, *Terminal coalgebras in well-founded set theory*, Theoretical Comp. Sci. **114** (1993), 299–315.

[Bec67] J.M. Beck, *Triples, algebras, and cohomology*, Ph.D. thesis, Columbia University, 1967.

[BH76] B. Banaschewski and H. Herrlich, *Subcategories defined by implications*, Houston Journal of Mathematics **2** (1976), no. 2.

[Bir35] G. Birkhoff, *On the structure of abstract algebras*, Proceedings of the Cambridge Philosophical Society **31** (1935), 433–454.

[Bir44] G. Birkhoff, *Subdirect unions in universal algebra*, Bulletin of the American Mathematical Society **50** (1944), 764 – 768.

[BM96] Jon Barwise and Lawrence Moss, *Vicious circles*, CSLI Lecture Notes, vol. 60, Stanford University, 1996.

[Bor94] F. Borceux, *Handbook of categorical algebra I-III*, Cambridge University Press, 1994.

[But98] Carsten Butz, *Regular categories and regular logic*, BRICS LS-98-2, October 1998, Course notes, 35 pp.

[BW82] Stephen Bloom and Jesse B. Wright, *Finitary quasi-varieties*, Journal of Pure and Applied Algebra **25** (1982), 121–154.

[BW85] Michael Barr and Charles Wells, *Toposes, triples and theories*, Grundlehren der maathematischen Wisenschaften, vol. 278, Springer-Verlag New York, 1985.

[Che80] Brian F. Chellas, *Modal logic: An introduction*, Cambridge University Press, 1980.

[Cîr00] Corina Cîrstea, *An algebra-coalgebra framework for system specification*, In Reichel [**Rei00**], pp. 81–112.

[CLM01] A. Corradini, M. Lenisa, and U. Montanari (eds.), *Electronic notes in theoretical computer science*, vol. 44, Coalgebraic Methods in Computer Science (CMCS'2001), 2001.

[Coc93] J. R. B. Cockett, *Introduction to distributive categories*, Mathematical Structures in Computer Science **3** (1993), no. 3, 277–307.

[Cro93] Roy L. Crole, *Categories for types*, Cambridge University Press, 1993.

[DP90] B. A. Davey and Hilary Priestley, *Introduction to lattices and order*, Cambridge University Press, Cambridge, 1990.

[EM65] S. Eilenberg and J. C. Moore, *Adjoint functors and triples*, Illinois Journal of Mathematics **9** (1965), 381–398.

[FS79] M. P. Fourman and D. S. Scott, *Sheaves and logic*, Applications of Sheaf Theory to Algebra, Analysis, and Topology (M. P. Fourman, C. J. Mulvey, and D. S. Scott, eds.), Lecture Notes in Mathematics, Springer-Verlag, 1979.

[FS90] P. Freyd and A. Scedrov, *Categories, allegories*, North-Holland, Amsterdam, 1990.

[GHS01] H. Peter Gumm, Jesse Hughes, and Tobias Schröder, *Distributivity of categories of coalgebras*, FG Informatik 25, Philipps-Universität Marburg, 2001.

[Gol99] Robert Goldblatt, *What is the coalgebraic analogue of Birkhoff's Variety Theorem?*, Available at `http://www.mcs.vuw.ac.nz/~rob/`, December 1999.

[Grä68] George Grätzer, *Universal algebra*, D. Van Nostrand Company, Inc., 1968.

[GS98] H. Peter Gumm and Tobias Schröder, *Covarieties and complete covarieties*, In Jacobs et al. [**JMRR98**], pp. 43–56.

[GS01] H. Peter Gumm and Tobias Schröder, *Products of coalgebras*, Algebra Universalis (2001), To appear.

[Gum98] H. Peter Gumm, *Equational and implicational classes of coalgebras*, 1998, Extended Abstract submitted for RelMiCS'4.

[Gum99] H. Peter Gumm, *Elements of the general theory of coalgebras*, LUATCS'99, Rand Africaans University, Johannesburg, South Africa. Available at `http://proof.mth.uct.ac.za/~pjipsen/luatcs99/`, 1999.

[Gum01a] H. Peter Gumm, *Birkhoff's variety theorem for coalgebras*, Contributions to General Algebra 2000 (2001), To appear.

[Gum01b] H. Peter Gumm, *Functors for coalgebras*, Algebra Universalis (2001), To appear.

[HC68] G. Hughes and M. Cresswell, *An introduction to modal logic*, Methuen, London, 1968.

[HJ99] Ulrich Hensel and Bart Jacobs, *Coalgebraic theories of sequences in PVS*, Journal of Logic and Computation **9** (1999), no. 4, 463–500.

[Hug00] Jesse Hughes, *The coalgebraic analogue of Birkhoff's variety theorem (abstract)*, Bulletin of Symbolic Logic, June 2000, Abstract for talk delivered at 1999–2000 Winter Meeting of the ASL, Washington, DC.

[Hug01] Jesse Hughes, *Modal operators for coequations*, In Corradini et al. [**CLM01**].

[Jac95] B. Jacobs, *Mongruences and cofree coalgebras*, Lecture Notes in Computer Science **936** (1995).

[Jac99] B.P.F. Jacobs, *The temporal logic of coalgebras via Galois algebras*, Tech. Report CSI-R9906, Computing Science Institute, April 1999.

[Jac00] Bart Jacobs, *Towards a duality result in the modal logic of coalgebras*, In Reichel [**Rei00**], pp. 163–198.

[JMRR98] Bart Jacobs, Larry Moss, Horst Reichel, and Jan Rutten (eds.), *Electronic notes in theoretical computer science*, vol. 11, Coalgebraic Methods in Computer Science (CMCS'1998), 1998.

[JPT$^+$98] Peter Johnstone, John Power, Toru Tsujishita, Hiroshi Watanabe, and James Worrell, *The structure of categories of coalgebras*, August 1998.

[JR97] Bart Jacobs and Jan Rutten, *A tutorial on (co)algebras and (co)induction*, Bulletin of the European Association for Theoretical Computer Science **62** (1997).

[JR99] Bart Jacobs and Jan Rutten (eds.), *Electronic notes in theoretical computer science*, vol. 19, Coalgebraic Methods in Computer Science (CMCS'1999), 1999.

[Kle65] H. Kleisli, *Every standard construction is induced by a pair of adjoint functors*, Proceedings of the American Mathematical Society **16** (1965), 544–546.

[Kur99] Alexander Kurz, *Modal rules are co-implications*, Draft. Available at `http://helios.pst.informatik.uni-muenchen.de/~kurz`, February 1999.

[Kur00] Alexander Kurz, *Logics for coalgebras and applications for computer science*, Ph.D. thesis, Ludwig-Maximilians-Universität München, 2000.

[Lam70] Joachim Lambek, *Sub equalizers*, Canadian Mathematical Bulletin **13** (1970).

[Lan71] S. Mac Lane, *Categories for the working mathematician*, Graduate Texts in Mathematics, Springer-Verlag, 1971.

[Law63] Bill Lawvere, *Functorial semantics of algebraic theories*, Proceedings of the National Academy of Sciences of the United States of America **50** (1963).

[LM92] Saunders Mac Lane and Ieke Moerdijk, *Sheaves in geometry and logic: a first introduction to topos theory*, Universitext, Springer-Verlag, 1992.

[LS86] J. Lambek and P. J. Scott, *Introduction to higher order categorical logic*, Cambridge University Press, 1986.

[Man76] Ernie Manes, *Algebraic theories*, Graduate Texts in Mathematics, no. 26, Springer-Verlag, 1976.

[Maš01] Dragan Mašulović, *Invariants of monadic coalgebras*, In Corradini et al. [**CLM01**].

[Mos99] Lawrence S. Moss, *Coalgebraic logic*, Annals of Pure and Applied Logic **96** (1999), 277–317.

[MT92] K. Meinke and J. V. Tucker, *Universal algebra*, In Abramsky et al. [**AGM92**], pp. 189–412.

[Ném82] I. Németi, *On notions of factorizations systems and their applications to cone-injective subcategories*, Periodica Mathematica Hungarica **13** (1982), no. 3, 229–235.

[PZ01] Erik Poll and Jan Zwanenburg, *From algebras and coalgebras to dialgebras*, In Corradini et al. [**CLM01**].

[Rei00] H. Reichel (ed.), *Electronic notes in theoretical computer science*, vol. 33, Coalgebraic Methods in Computer Science (CMCS'2000), 2000.

[Rob94] Edmund Robinson, *Variations on algebra: Monadicity and generalisations of equational theories*, 1994.

[Roş01] Grigore Roşu, *Equational axiomatizability for coalgebra*, Theoretical Computer Science **260** (2001), Scheduled to appear.

[RTJ01] J. Rothe, H. Tews, and B. Jacobs, *The coalgebraic class specification language ccsl*, Journal of Universal Computer Science (JUCS) **7** (2001), no. 2, 175–193.

[Rut96] J.J.M.M. Rutten, *Universal coalgebra: a theory of systems*, Tech. Report CS-R9652, Centrum voor Wiskunde en Informatica, 1996.

[Tay99] Paul Taylor, *Practical foundations of mathematics*, Cambridge University Press, 1999.

[Tur96] Daniele Turi, *Functorial operational semantics and its denotational dual*, Ph.D. thesis, Free University, Amsterdam, June 1996.

[vK01] Burghard von Karger, *Temporal algebra*, Mathematical Structures in Computer Science (2001), To appear.

[Wor98] James Worrell, *Toposes of coalgebras and hidden algebras*, In Jacobs et al. [**JMRR98**].

[Wor00] James Worrell, *Terminal sequences for accessible endofunctors*, A chapter of Worrell's dissertation, 2000.

Index

absolute coequalizer, 55
accessibility relation, 71
algebra
– for a monad, 48–51
– for an endofunctor, 7
coequalizers of –, 78
colimits of –, 19–22, 77–89
coproducts of –, 89
final, 32
free, 47, 51–53, 115, 133
homomorphism, 7
initial, 32–40, 45–46
limits of –, 16–18
minimal, 40
universal, 10–11, 36–37, 39
almost co-regular, 21
almost regular, 21
associativity condition, 217
automata, 108, 146, 164

Beck's theorem, 55
behavioral covariety, 168–172
Birkhoff variety, 115, 116
Birkhoff variety theorem, 122
Bisim, 91
bisimilar, 105–108, 196
bisimulation, 92, 197
total, 168–170
bounded functor, 160

carrier, 7, 8
category
almost co-regular, 21
almost regular, 21
co-Birkhoff, 141
exact, 216
extensive, 175–176
quasi-co-Birkhoff, 140, 141
regular, 216
child, 14

choice
axiom of –, 90, 92, 93, 105, 197, 212
weak axiom of –, 31, 212
closed
– under codomains of epis, 141
– under coproducts, 141
– under products, 114
– under quotients, 114
– under regular subobjects, 141
– under subobjects, 114
deductively –, 136, 137
co-well-powered, 213
coalgebra
– for a comonad, 48–51, 150
– for an endofunctor, 8
cofree, 57, 58
colimits of –, 16–18
equalizers of –, 78
final, 32–34, 41–43, 45–46
homomorphism, 8
initial, 32
limits of –, 19–22, 77–89
products of –, 79–89
simple, 43–45, 106, 196
coequalizer
absolute, 55
split, 55
coequation, 140, 143–145, 149, 162, 166, 171, 198, 200, 201, 203
conditional, 149–153
conditional –, 149, 150, 152, 199, 200, 206
endo-invariant, 162–168
generating –, 163
cofree, 57
cofree coalgebra, 57, 58
coinduction, 43–45, 105–107, 195–197
coloring, 57
comonad, 217
comonadic, 59
complete Heyting algebra, 72

congruence, 29–32
conjunct covering, 154
conjunct representation, 155
conjunctly irreducible, 154
conjunctly presentable, 155
constructibly equivalent, 173
context, 181
corecursion, 41–43
cotriple, 217
covarietor, 59
covariety, 141
  behavioral, 168–172

deductively closed, 136, 137
diagonal, 215
distributive, 155

$\mathcal{E}^{\Gamma}$, 8
  colimits in –, 19–22, 77–89
  coproducts in –, 89
  limits in –, 16–18
$\mathcal{E}_{\Gamma}$, 8
  colimits in –, 16–18
  equalizers in –, 78
  limits in –, 19–22, 77–89
  products in –, 79–89
Eilenberg-Moore theorem, 49
elementary variety, 173
endomorphism-invariant, 162–168
endomorphism-stable, 136
epi
  – splits, 212
  regular, 209
epi-regular mono factorizations, 211
$\varepsilon^{\mathbf{V}}$, 121
equality, 215
equation, 115, 117–119, 134
equational domain, 127
equational variety, 115, 118
equivalence relation, 216
  effective –, 216
$\eta^{\mathbf{V}}$, 119
exact
  category, 216
  sequence, 22
extensive, 155

$F^{\mathbf{V}}$, 119
factorization system, 209–212
  epi-regular mono, 211
  regular epi-mono, 211
  regular epi-regular mono, 177
  strong epi-mono, 212
filter, 15
final, 32
finitely presentable, 128
fixed points, 32–34, 117, 139
forgetful functor, 8
foundation
  axiom of, 40
free algebra, 47, 51, 52, 115, 133
functor
  – creates colimits, 17
  – creates limits, 17
  – nearly preserves pullbacks, 176
  – preserves colimits, 17
  – preserves intersections, 74, 167
  – preserves limits, 16
  – preserves regular relations, 101
  – reflects colimits, 17
  – reflects limits, 17
  bounded, 160
  comonadic, 59
  monadic, 55

$\mathbb{G}$-coalgebra, 49
Galois correspondence, 119
Γ-algebra, 7
Γ-coalgebra, 8
graph, 92, 169, 215

Heyting algebra, 69, 72, 73, 101, 103, 151, 212
  complete, 72
homomorphism, 7, 8
Hughes, Quincy, iii

induction, 37–40, 53
inductive predicate, 38
initial, 32
intersection
  preservation of –, 74
invariant coequation, 162–168

join, 213
jointly monic, 214

Kripke frames, 15
Kripke models, 14–15, 70–72, 77
Kripke structures, 15
Kripke-Joyal semantics, 198–200

Lambek's lemma, 32, 33, 45
lifting functor, 36, 41
locally finitely presentable, 128
logos, 175, 177, 178

locally complete, 175, 177, 178, 180–185, 198

minimal, 38
$\mathsf{Mod}_{\mathfrak{A}}(\phi)$, 71
monad, 216–217
monadic, 55
mono
– splits, 212
regular, 209
Mon, 122

$\mathbb{N}$, 34–36, 39
$\overline{\mathbb{N}}$, 41
natural numbers object, 34
nearly preserves pullbacks, 176
neighborhood, 16
neighborhood filter, 16
NNO, 34
notation conventions, 209
NWF, 34

object oriented languages, 60, 146
ω, 36
$\overline{\omega}$, 41
opposite relation, 215
orthogonality, 118–119

$\mathbb{P}$, 33
$\perp$ (perp), 118–119, 141
pointwise semantics, 200–203
polynomial functor, 43
coalgebras for -, 76
power set functor, 33
pre-congruence, 27–29
predicates, 212–214
pullbacks
nearly preserves –, 175–177, 186
weak –, 44, 45, 93, 102, 108, 196

quasi-Birkhoff variety, 116
quasi-co-Birkhoff category, 140
quasi-covariety, 141
quasi-variety, 116, 119–122
quasi-variety theorem, 119

recursion, 34–37, 53
reflexive, 215
RegSub, 63
RegSub, 214
regular
category, 216
epi, 209
mono, 209
regular co-relation, 104
regular epi-mono factorizations, 211
regular epi-reflective, 119
regular n-ary relation, 109
regular relation, 90, 92, 99, 101–103
regularly co-well-powered, 213
regularly well-powered, 213
relation, 214–216
preserves regular -, 101
regular –, 90, 92, 99, 101–103
retractable presentation, 129
retractably presentable, 130

$\mathcal{S}at$, 135
SET, 33
non-well-founded, 34
well-founded, 34, 39–40
Set
quasi-varieties in, 122
S4, 69, 100
S4, 47, 69, 110, 164, 165, 190, 199
Σ-algebra, 10
signature, 10
specification
coalgebraic, 60
coalgebraic, 146, 148
split coequalizer, 55
stable, 136, 140
– under pullbacks, 216
streams, 13–14, 42–43, 107
finite, 11–12, 37, 39
infinite, 12–13, 41–42, 76
strong epi, 209
strong mono, 210
structure map, 7, 8
Sub, 212
SubAlg, 23–26
subalgebra, 23
generated by P, 24
subcoalgebra, 62
cogenerated by P, 64, 69
generated by P, 73, 77
subdirectly irreducible, 154
subobject classifier, 179–180
subobjects, 212–214
closed under α, 38
successor functor, 34, 41
symmetric, 215

$\mathbb{T}$-algebra, 48
$T^{\mathbf{V}}$, 121

terminal, 32
Top, 15–16
total bisimulation, 168–170
total relation, 168
transitive, 216
trees, 14, 36–37, 39, 43, 107
  well-founded, 39
triple, 216

U, 8, 18
  creates (co)limits, 18
$U^{\mathbf{V}}$, 119
uniformly Birkhoff, 127
uniformly co-Birkhoff, 153
union, 213
unit condition, 217

valid, 183
varietor, 56
variety, 115, 116
  Birkhoff, 116
  Birkhoff –, 115
  elementary –, 173
  equational –, 115
  quasi-Birkhoff –, 116

weak pullbacks, 44, 45, 93, 102, 108, 196
well-founded, 39–40
well-pointed, 159
well-powered, 213
WF, 34