Errata for *Admissible Digit Sets*

J. Hughes, M. Niqui / Theoretical Computer Science 351 (2006) 61–73

This is a correct version of Lemma 4.2 (p. 69) that fills a gap. The new parts are in red.

Lemma 4.2 *Let* $A \in \mathbb{M}$ *such that* $\delta(A, \overline{\mathbb{R}^+}) < \mathsf{red}(\Phi)$. *Then there exists* $0 \leq i < l$ *such that* $A \sqsubseteq \phi_i$.

Proof. By (3.2.b), we know that $A(\mathbb{R}^+) \subseteq \bigcup_{i=0}^{l-1} \phi_i(\mathbb{R}^+)$ where $A(\mathbb{R}^+)$ and each $\phi_i(\mathbb{R}^+)$ are intervals. Hence, either there is an i such that $\phi_i(0) \in A(\mathbb{R}^+)$ or there is an i such that $A(\mathbb{R}^+) \subseteq \phi_i(\mathbb{R}^+)$. In the latter case, we have $A \sqsubseteq \phi_i$.

For the former case, suppose there is an $x \in A(\mathbb{R}^+)$ such that $x = \phi_i(0)$ for some $\phi_i \in \Phi$. Since Φ is finite without loss of generality we can assume that x is the smallest number with this property, i.e.,

$$\forall i' < k, \phi_{i'}(0) < \phi_i(0) \Rightarrow \mathbf{S}_0\Big(\phi_{i'}(0)\Big) < \mathbf{S}_0\Big(A(0)\Big) \ . \qquad (4.1)$$

By assumption, $\delta(A, \overline{\mathbb{R}^+}) < \mathsf{red}(\Phi)$ and so $\mathbf{S}_0(A(+\infty)) - \mathbf{S}_0(A(0)) < \mathsf{red}(\Phi)$, where $\mathbf{S}_0 \colon \overline{\mathbb{R}^+} \longrightarrow [-1, 1]$ is the bijection from Section 2. It follows that

$$\begin{aligned} \mathbf{S}_0(A(+\infty)) - \mathbf{S}_0(x) &< \mathsf{red}(\Phi) \ , \\ \mathbf{S}_0(x) - \mathbf{S}_0(A(0)) &< \mathsf{red}(\Phi) \ , \end{aligned}$$

so $\mathbf{S}_0(A(\mathbb{R}^+)) \subsetneq [\mathbf{S}_0(x) - \mathsf{red}(\Phi), \mathbf{S}_0(x) + \mathsf{red}(\Phi)]$. Since $x \in \mathbb{R}^+$, there is a $\phi_j \in \Phi$ with $x \in \phi_j(\mathbb{R}^+)$. Note that $\phi_j(0) < \phi_i(0)$ and hence by (4.1) $\mathbf{S}_0\Big(\phi_j(0)\Big) < \mathbf{S}_0\Big(A(0)\Big)$. Moreover, minimality of $\mathsf{red}(\Phi)$ in (4.4) means that the end points $\mathbf{S}_0(\phi_j(0))$ and $\mathbf{S}_0(\phi_j(+\infty))$ are at least at a distance $\mathsf{red}(\Phi)$ from $\mathbf{S}_0(x)$. In other words,

$$[\mathbf{S}_0(x) - \mathsf{red}(\Phi), \mathbf{S}_0(x) + \mathsf{red}(\Phi)] \subseteq [\mathbf{S}_0(\phi_j(0)), \mathbf{S}_0(\phi_j(+\infty))] \ ,$$

and so $A \sqsubseteq \phi_j$. □

This is a correct version of Theorem 4.3 (pp. 69–70). The new parts are in red.

Theorem 4.3 *Let $A \in \mathbb{M}$ and $\alpha \in \Phi^\omega$ be given and let $\beta = \bigsqcup \mathsf{em}^{A,\alpha}(t)$. Then $\beta \in \Phi^\omega$.*

Proof. We prove by induction that, for every j, there exists an n such that $\mathsf{length}(\mathsf{em}^{A,\alpha}(n)) \geq j$. The base case ($j = 0$) is trivial. For the inductive step, we will suppose that the claim is true for some j and prove it for $j + 1$.

Let n be given, then, such that $\mathsf{length}(\mathsf{em}^{A,\alpha}(n)) \geq j$ and let B be the matrix of coefficients for $\mathsf{M}^{A,\alpha}(n)$. Let

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} , \quad M = \begin{bmatrix} \frac{1}{b_{11}+b_{21}} & \frac{1}{b_{12}+b_{22}} \end{bmatrix} .$$

By induction, B is refining, so $b_{11} + b_{21} \neq 0$ and $b_{12} + b_{22} \neq 0$ (see comments following Equation (4.2)).

Let

$$X := \frac{1}{\max(|M(0)|, |M(+\infty)|)} . \tag{4.2}$$

Since $\lim_{j\to\infty} \mathcal{B}(\Phi, j) = 0$, there exists N such that

$$\mathcal{B}(\Phi, N) < \frac{\mathsf{red}(\Phi)\ X^2}{|\det B|} . \tag{4.3}$$

Take $J = n + N + 1$. We claim that $\mathsf{length}(\mathsf{em}^{A,\alpha}(J)) \geq j + 1$. Let $\alpha = \phi_{i_0}\phi_{i_1}\phi_{i_2}\ldots$ and let

$$C = \phi_{i_{\mathsf{ab}^{A,\alpha}(n)}} \circ \phi_{i_{\mathsf{ab}^{A,\alpha}(n)+1}} \circ \ldots \circ \phi_{i_{\mathsf{ab}^{A,\alpha}(n)+N-1}} .$$

The Möbius map C, then, is constructed by taking the composition of the next N digits of the input stream α.

We may assume that every step from n to $n + N$ (inclusive) is an absorption step, so that

$$h^{A,\alpha}(J - 1) = \langle B \circ C, \mathsf{em}^{A,\alpha}(n), \mathsf{ab}^{A,\alpha}(n) + N\rangle .$$

Now, by our choice of N, we have

$$\delta(C, \mathbb{R}^+) < \frac{\mathsf{red}(\Phi)\ X^2}{|\det B|} .$$

We calculate

$$\begin{aligned}
\delta(B \circ C, \mathbb{R}^+) &= \delta(B, C(\mathbb{R}^+)) \\
&= \delta(C, \mathbb{R}^+) \cdot |\det B| \cdot M(C(0)) \cdot M(C(+\infty)) && \text{by (2.2)} \\
&\leq \delta(C, \mathbb{R}^+) \cdot \frac{|\det B|}{X^2} && \text{by (4.2)} \\
&< \mathsf{red}(\Phi) \ .
\end{aligned}$$

Hence we can apply Lemma 4.2 and obtain ϕ_i such that $B \circ C \sqsubseteq \phi_i$. Thus, we see that the Jth step is an emission step, as desired. □