FUZZY DYNAMIC LOGIC

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Abstract. We introduce fuzzy set semantics for dynamic logic in order to provide a sense of efficacy for means-end semantics. We also discuss an alternative interpretation of multimodal logic in terms of fuzzy sets. *This is a working draft.*

1. Introduction

We investigate here a fuzzy set semantics for propositional dynamic logic. This semantics differs significantly from other interpretations of modal operators using fuzzy sets. In particular, we have chosen to interpret the dynamic structure in terms of expected outcomes rather than in terms of standard fuzzy relations. This decision is the distinguishing feature of our approach and is motivated by our aim of providing a semantics for means-end relations.

For our purposes, a means-end relation is a proposition asserting the adequacy of some action α in bringing about a condition φ. The action α is said to be a *means* to the *end* φ. This interpretation of means-end relations is somewhat presumptuous, strongly suggesting the dynamic logic that we investigate here. A fuller account of the semantics of means-end relations requires a justification of this initial step, that means are actions and ends are propositions and a means-end relation "α is a means to φ," just asserts that α is adequate in bringing about φ. We will not address these philosophical questions here and in fact our account of means-end relations here is admittedly naive. In later work, we will more fully justify a modification of the dynamic logic we consider here, one in which a conditional operator is used to represent the usually implicit preconditions of means-end relations. For now, we content ourselves with *local* means-end relations: propositions that *in this world*, doing α is adequate to bring about φ.

With these semantic concerns aside for now, we turn to a natural formalism for representing such means-end statements, namely *propositional dynamic logic (PDL)*: a propositional logic with a family of weak modal operators $\langle\alpha\rangle$ for each action α. The standard interpretation of $\langle\alpha\rangle\varphi$ is essentially just what we need: if one does α, then φ can result. But here one must be careful about the interpretation of "can". The standard models of PDL involve classical non-determinism: for each world w and action α, the outcome $[\![\alpha]\!](w)$ of doing α in w is a *set* of worlds which is usually interpreted as the set of worlds which may result. If the non-determinism represents a choice on the part of an agent then $\langle\alpha\rangle$ has the approximately correct interpretation for means-end relations, but not if the non-determinism includes factors outside of the control of our agent (say, randomness or unknown environmental variables).

Mark Brown presents a semantics that avoids these problems in [2]. He uses minimal models to reason about one's ability to bring about an end, but he is not concerned with the means per se. Brown's "can" operator $\diamondsuit$ and our dynamic operators are related as follows: one *can* bring about φ just in case there is a means α to φ — issues of fuzziness aside. In Section 5, we present a fuzzy set semantics very closely related to Brown's logic of ability.

But neither Brown nor standard PDL semantics include a feature that we consider essential to standard means-end relations. Namely, different means to the same end may be compared according to their *efficacy*, or propensity to bring about the end. The outcome of an action is often unknown, but rough probabilities can often be given regarding the outcome (at least in our idealized models of possible worlds, if not in real life). A darts expert has a greater chance of hitting the double bull as a result of throwing his dart than the darts novice, flipping a biased coin gives a different probability

of heads than flipping an unbiased coin, and so on. Among the various comparisons between means, efficacy is surely one of the most important but it does not arise in models involving essentially non-deterministic transitions.

Of course, probabilistic transition systems are much studied in computer science, where one uses *Probabilistic Computation Tree Logic* (pCTL) to reason about and characterize such structures [9, 1]. pCTL is a rich language, including "unless" and "until" operators that could be useful in our setting (and may be considered in later work). However, pCTL is aimed at probabilities in temporal, not dynamic, logic and consequently does not provide an appropriate composition of operators. We need to be able to assert that φ has probability at least x if one first does α and then β, but this cannot be expressed with a single formula involving pCTL's probabilistic operators.

Instead, we turn to fuzzy sets for our semantics. We interpret $\langle\alpha\rangle\varphi$ as the vague proposition "α is a reliable means to φ." The interpretation of $\langle\alpha\rangle\varphi$ is based on the probability that the outcome of doing α will be a world which realizes φ. In this respect, we are following the program of using the probability of a formula to define the truth degree of a fuzzy predicate outlined in [8] (but we choose different interpretations of our connectives).

2. Propositional dynamic logic

Propositional Dynamic Logic (PDL) is a logic of actions, typically used to reason about computer program behavior. We refer the reader to [10], from which we take much of the following material. The syntax consists of two disjoint types: the set Π_0 of atomic actions and the set Φ_0 of atomic propositions. From these two sets, we inductively define the sets Π of actions and Φ of formulas as follows

- $\{\top\} \cup \Phi_0 \subseteq \Phi$;
- if $\varphi, \psi \in \Phi$ then $\neg\varphi$ and $\varphi \wedge \psi \in \Phi$;
- if $\alpha \in \Pi$ and $\varphi \in \Phi$ then $[\alpha]\varphi \in \Phi$;
- $\Pi_0 \subseteq \Pi$;
- if $\alpha, \beta \in \Pi$ then $(\alpha; \beta)$, $(\alpha \cup \beta)$ and α^* are in Π;
- if $\varphi \in \Phi$ then $\varphi? \in \Pi$.

We introduce the value $\bot$, the connectives $\vee$, $\rightarrow$ and $\leftrightarrow$ and the weak operator $\langle\alpha\rangle$ as usual. The action constructors are intended thus: the semicolon denotes sequential composition, the union non-deterministic choice and the asterisk finite iterations.

A *PDL frame* $\mathbf{F}$ for Π_0 consists of a set $\mathcal{W}$ of worlds (or states) and an interpretation $[\![-]\!]^{\mathbf{F}} : \Pi_0 \rightarrow (\mathcal{P}\mathcal{W})^{\mathcal{W}}$ of actions via non-deterministic transition systems. Here $\mathcal{P}$ denotes the powerset functor and exponentiation A^B denotes the set of functions $B \rightarrow A$.

The interpretation $\Pi_0 \rightarrow (\mathcal{P}\mathcal{W})^{\mathcal{W}}$ assigns to each $m \in \Pi_0$ a function $[\![m]\!] : \mathcal{W} \rightarrow \mathcal{P}\mathcal{W}$. For $w \in \mathcal{W}$, we interpret $[\![m]\!](w) \subseteq \mathcal{W}$ as the set of possible[1] outcomes of doing m in w. Clearly, a PDL frame is just the same as a labeled transition system with nodes $w \in \mathcal{W}$ and labels $m \in \Pi_0$.

A *PDL model* $\mathbf{M}$ is a frame $\mathbf{F}$ together with a valuation $[\![-]\!]^{\mathbf{M}} : \Phi_0 \rightarrow \mathcal{P}\mathcal{W}$ of atomic propositions. We abuse notation by adopting Scott brackets for both the valuation of atomic propositions and the interpretation of atomic actions, but since our sets Π_0 and Φ_0 are disjoint, no confusion should result. We omit the superscripts when possible.

A valuation assigns to each atomic proposition $P \in \Phi_0$ a set $[\![P]\!] \subseteq \mathcal{W}$ of worlds. We interpret $[\![P]\!]$ as the set of worlds in which P is true.

We extend the valuation of atomic propositions to a function $[\![-]\!] : \Phi \rightarrow \mathcal{P}\mathcal{W}$ and the interpretation of atomic actions to a function $[\![-]\!] : \Pi \rightarrow (\mathcal{P}\mathcal{W})^{\mathcal{W}}$ recursively as in Table 1. We say that w *satisfies* φ or that φ *is true* in w just in case $w \in [\![\varphi]\!]$. In this case, we write $\mathbf{M}, w \models \varphi$ or just $w \models \varphi$ when $\mathbf{M}$ is understood by context. We write $\mathbf{M} \models \varphi$ and say that φ is *valid in* $\mathbf{M}$ if for every $w \in \mathcal{W}$ we have $w \models \varphi$ and we write $\models \varphi$ if $\mathbf{M} \models \varphi$ for every model $\mathbf{M}$. In this case, we say that φ is *valid*.

[1] Or "normal" or "reasonably expected" or ...

On formulas

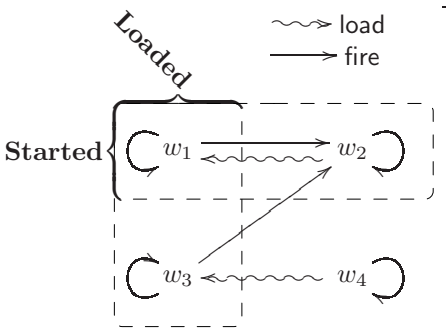
$$[\![\top]\!] = \mathcal{W}$$
$$[\![\neg\varphi]\!] = \mathcal{W} \setminus [\![\varphi]\!]$$
$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$$
$$[\![[\alpha]\varphi]\!] = \{w \in \mathcal{W} \mid [\![\alpha]\!](w) \subseteq [\![\varphi]\!]\}$$

On actions

$$[\![\alpha;\beta]\!](w) = \{w' \in \mathcal{W} \mid \exists w'' \in \mathcal{W} \,.\, w'' \in [\![\alpha]\!](w) \text{ and } w' \in [\![\beta]\!](w'')\}$$
$$[\![\alpha \cup \beta]\!](w) = [\![\alpha]\!](w) \cup [\![\beta]\!](w)$$
$$[\![\alpha^*]\!](w) = \bigcup_{n=0}^{\infty} [\![\underbrace{\alpha;\ldots;\alpha}_{n \text{ times}}]\!](w)$$
$$[\![\varphi?]\!](w) = \begin{cases} \{w\} & \text{if } w \in [\![\varphi]\!]; \\ \emptyset & \text{else.} \end{cases}$$

TABLE 1. Extension of valuation to Φ and interpretation to Π.

Let us call an action α *prohibited in* w if $[\![\alpha]\!](w) = \emptyset$. Intuitively, such actions cannot be performed in w. If α is prohibited in w, then $w \models [\alpha]\varphi$ for any $\varphi \in \Phi$ (including $\bot$), but $w \not\models \langle\alpha\rangle\varphi$ for any $\varphi \in \Phi$ (not even $\top$).



Sample facts about our model

$$[\![\mathbf{Started}]\!] = \{w_1, w_2\}$$
$$[\![\mathbf{Loaded}]\!] = \{w_1, w_3\}$$
$$[\![[\mathsf{fire}]\mathbf{Started}]\!] = [\![\mathbf{Started}]\!]$$
$$[\![\langle\mathsf{fire}\rangle\mathbf{Started}]\!] = [\![\mathbf{Started}]\!] \cup [\![\mathbf{Loaded}]\!]$$
$$[\![[\mathsf{load}]\mathbf{Loaded}]\!] = \mathcal{W}$$
$$[\![\langle\mathsf{load}\rangle\mathbf{Loaded}]\!] = [\![\neg\mathbf{Loaded}]\!]$$
$$[\![[(\mathsf{load} \cup \mathsf{fire})^*]\mathbf{Started}]\!] = [\![\mathbf{Started}]\!]$$
$$[\![\langle(\mathsf{load} \cup \mathsf{fire})^*\rangle\mathbf{Started}]\!] = \mathcal{W}$$
$$[\![[\mathbf{Loaded}?;\mathsf{fire}]\mathbf{Started}]\!] = [\![\mathbf{Started}]\!] \cup [\![\neg\mathbf{Loaded}]\!]$$
$$[\![\langle\mathbf{Loaded}?;\mathsf{fire}\rangle\mathbf{Started}]\!] = [\![\mathbf{Loaded}]\!]$$

FIGURE 1. A sample PDL model

Example 2.1. Consider the example of a footrace about to begin. The starter has a (one-shot) pistol and the race will begin as soon as he pulls the trigger *provided* the pistol is loaded. We will construct a very simple model for this case consisting of only two atomic predicates:

Started	true if the race has started,
Loaded	true if the pistol is loaded.

Our language will also include two atomic actions:

load	the starter loads the pistol,
fire	the starter pulls the trigger.

Note that the action fire does not imply that the pistol discharges a blank, but only that the starter pulls the trigger. Our action name fire may be a bit misleading in this respect, but it is more suggestive than pull and less awkward than pulltrigger.

We consider a model of four worlds, so that each combination of atomic predicates is represented. See Figure 1, in which an arrow $w \rightsquigarrow w'$ denotes that $w' \in [\![\mathsf{load}]\!](w)$ and $w \longrightarrow w'$ that $w' \in [\![\mathsf{fire}]\!](w)$. The transition structure for loaded takes a world in which the pistol is unloaded to one in which it is loaded without altering the truth value for **Started**. The act is prohibited in worlds which already satisfy **Loaded**. You can't load an already loaded pistol.

For worlds in which **Loaded** is false, the action fire produces no change. Successfully firing the pistol in the world w_1 in which the race has already begun moves one to w_2, in which the state of **Started** is unchanged, but **Loaded** is false. Successfully firing the pistol in world w_3 not only unloads the gun, but also starts the race, so again we have a transition to w_2.

Just to make our model properly non-deterministic, we add the possibility that a blank misfires, say, due to a weak spring in the pistol. In this case, the blank is still capable of firing but our hammer has not struck it with enough force to fire it. Hence, we add reflexive fire-transitions in worlds w_1 and w_3.

The reader may confirm that the equations in Figure 1 are actually satisfied by the model.

To complete our introduction to PDL, we present in Table 2 the standard axiom system for PDL, taken from [10]. We omit the proof that this system is sound and complete for our semantics, i.e. that $\vdash \varphi$ iff $\models \varphi$.

Axioms

Tautology	Every propositional tautology
Distributivity	$[\alpha](\varphi \wedge \psi) \leftrightarrow ([\alpha]\varphi \wedge [\alpha]\psi)$
Composition	$[\alpha;\beta]\varphi \leftrightarrow [\alpha][\beta]\varphi$
Choice	$[\alpha \cup \beta]\varphi \leftrightarrow ([\alpha]\varphi \wedge [\beta]\varphi)$
Iteration	$[\alpha^*]\varphi \leftrightarrow (\varphi \wedge [\alpha][\alpha^*]\varphi)$
Test	$[\psi?]\varphi \leftrightarrow \psi \rightarrow \varphi$
Induction	$[\alpha^*](\varphi \rightarrow [\alpha]\varphi) \rightarrow (\varphi \rightarrow [\alpha^*]\varphi)$
K	$[\alpha](\varphi \rightarrow \psi) \rightarrow ([\alpha]\varphi \rightarrow [\alpha]\psi)$

Inference rules

Modus Ponens	$\varphi, \varphi \rightarrow \psi / \psi$
Necessitation	$\varphi / [\alpha]\varphi$

TABLE 2. The theory PDL

We intend the language PDL to be our language for means-end relations. We close the section with a brief sketch of how this should go, but we postpone most of the philosophical justification for this sketch for another day.

A *local means-end relation* is an assertion that one may do α in *this* world in order to bring about φ. This is the narrow meaning of the statement "α is a means to φ". (A non-local relation would assert something like: in a "normal" world, doing α may yield φ.) Thus, a local means-end relation is naturally interpreted in PDL as $\langle\alpha\rangle\varphi$.

This is a weak means-end relation, of course. More often, when one says α is a (local) means to φ, one means

(1) if one does α, then φ will be realized and

(2) one *can* do α.

The second condition is naturally left implicit more often than not. Let us say that α is a *local strong means* to φ (in w) if conditions (1) and (2). In PDL, these conditions are equivalent to $w \models [\alpha]\varphi \wedge \langle\alpha\rangle\top$.

Semantically, α is a local means to φ in w iff $[\![\alpha]\!](w) \cap [\![\varphi]\!] \neq \emptyset$ and α is a local strong means to φ in w iff $\emptyset \subsetneqq [\![\alpha]\!](w) \subseteq [\![\varphi]\!]$.

3. Fuzzy sets and modal logics

We are interested in adding uncertainty to our analysis of means and ends. We want, however, to retain a truth-functional semantics and so we are attracted to the fuzzy logic approach to probabilities found in [8]. In this section, we present an introduction to fuzzy sets and also a new fuzzy modal logic and in the next, we show how to probabilistic interpretations dynamic logic as fuzzy sets.

The preliminary material on fuzzy sets can be found in greater detail in [11]. See also [7].

A fuzzy subset of a given set X is a function $S : X \to [0, 1]$. Given $x \in X$, the value $S(x)$ represents the degree to which x belongs in the fuzzy set S. Some authors prefer to distinguish between the fuzzy set S and its fuzzy characteristic function, usually denoted μ_S, but we do not do so here.

We define $\mathcal{F}X$ to be the set of fuzzy subsets of X. There is a natural injection $\mathcal{P}X \hookrightarrow \mathcal{F}X$ taking a subset $S \subseteq X$ to its characteristic function, which we also denote S, an abuse of notation consistent with our notation for fuzzy sets. We call such functions $S : X \to \{0, 1\}$ the *crisp* subsets of X. As an example, the set X itself is identified with the constant function $x \mapsto 1$.

We next define the set operations of union, intersection and complement. These will be functions of the form $\mathcal{F}X \times \mathcal{F}X \to \mathcal{F}X$ (for union and intersection) and $\mathcal{F}X \to \mathcal{F}X$ (for complement). It is desirable that these functions are induced by functions $u, i : [0, 1]\times[0, 1] \to [0, 1]$ and $c : [0, 1] \to [0, 1]$, resp., in the sense that

$$\begin{aligned}(S \cup T)(x) &= u(S(x), T(x)),\\ (S \cap T)(x) &= i(S(x), T(x)),\\ \overline{S}(x) &= c(S(x)).\end{aligned}$$

This will ensure that our operations are defined up-to-elementhood, so that if $S(x) = S(y)$ and $T(x) = T(y)$, then $(S \cup T)(x) = (S \cup T)(y)$ (and similarly for $\cap$ and $\neg$).

We will be primarily interested in the so-called *standard* operations, namely

$$\begin{aligned}(S \cup T)(x) &= \max\{S(x), T(x)\},\\ (S \cap T)(x) &= \min\{S(x), T(x)\},\\ \overline{S}(x) &= 1 - S(x).\end{aligned}$$

These operations are canonical in many senses and we do not pursue other interpretations here.

See [4, 7, 11] for discussion of other interpretations of union, intersection and negation.

Application of fuzzy set theory to modal logics dates back to [14]. Let us consider a propositional modal logic PML with an indexed set $\{\Box_i \mid i \in I\}$ of strong modal operators with corresponding conjugates $\{\Diamond_i \mid i \in I\}$. An indexed set of operators allows great flexibility in interpretation: in case we interpret the operators epistemically, then $\Box_i\varphi$ is typically interpreted as *agent* i knows φ. In the sense of necessity, the index set allows for varying strengths of necessity and relevance. Also, stated in this generality, one can view the propositional dynamic logic of Section 2 as a special case of PML — but, as we'll see in later sections, we choose a different fuzzy semantics for dynamic logic. In any case, the indexed operators of this section are merely formal; the index set is uninterpreted.

In classical modal logics, a *Kripke frame* consists of a set $\mathcal{W}$ of worlds and an interpretation $[\![-]\!] : I \to (\mathcal{P}\mathcal{W})^{\mathcal{W}}$. A *Kripke model* extends a frame by adding a valuation $[\![-]\!] : \Phi_0 \to \mathcal{P}\mathcal{W}$, which

extends to $\Phi \to \mathcal{P}\mathcal{W}$ via essentially the same definition as found in Table 1, minus the clauses for complex actions.

The corresponding fuzzy concepts result from replacing the powerset operation in the classical definition by the fuzzy subset operation. Thus, a *fuzzy frame* for PML is a set $\mathcal{W}$ of worlds together with an interpretation $[\![-]\!] : I \to (\mathcal{F}\mathcal{W})^{\mathcal{W}}$ and a *fuzzy model* is a fuzzy frame together with a valuation $[\![-]\!] : \Phi_0 \to \mathcal{F}\mathcal{W}$.

The (fuzzy) valuation $[\![P]\!]$ of an atomic proposition P gives a measure of the degree to which each world satisfies P. In other words, the greater $[\![P]\!](w)$, the greater degree to which w satisfies P (i.e. , the "more true" P is in w).

The interpretation $[\![i]\!]$ defines a fuzzy accessibility relation via the correspondence between $(\mathcal{F}\mathcal{W})^{\mathcal{W}}$ and $\mathcal{F}(\mathcal{W} \times \mathcal{W})$. This correspondence is the fuzzy analogue to the well known isomorphism $(\mathcal{P}\mathcal{W})^{\mathcal{W}} \cong \mathcal{P}(\mathcal{W} \times \mathcal{W})$. Explicitly, the value $[\![i]\!](w)(w') \in [0,1]$ gives the degree to which w' is i-accessible (or i-relevant) to w.

We extend our valuation to the set Φ of propositions of PML. This means we recursively define a function $[\![-]\!] : \Phi \to \mathcal{F}\mathcal{W}$, so that for each $w \in \mathcal{W}$ and $\varphi \in \Phi$, we have $[\![\varphi]\!](w) \in [0,1]$.

The extension for the boolean connectives $\wedge$, $\vee$ and $\neg$ is fairly obvious. We have already settled on the corresponding fuzzy set operations $\cap$, $\cup$ and $\setminus$ and so, as in the classical case, we define

$$\begin{aligned} [\![\varphi \wedge \psi]\!] &= [\![\varphi]\!] \cap [\![\psi]\!] \\ [\![\varphi \vee \psi]\!] &= [\![\varphi]\!] \cup [\![\psi]\!] \\ [\![\neg\varphi]\!] &= \mathcal{W} \setminus [\![\varphi]\!] \end{aligned}$$

This still leaves a choice for our interpretation of implication. There are traditionally two common choices, the S-implication and the R-implication. We have chosen the R-implication as the more interesting and better behaved notion of implication[2]: the property that conjunction is left adjoint to implication seems more fundamental than its definition involving $\neg$ and $\vee$.

S-implication:

$$\begin{aligned} [\![\varphi \to \psi]\!](w) &= [\![\neg\varphi \vee \psi]\!](w) \\ &= \max\{1 - [\![\varphi]\!](w), [\![\psi]\!](w)\} \end{aligned}$$

R-implication:

$$\begin{aligned} [\![\varphi \to \psi]\!](w) &= \sup\left\{x \in [0,1] \;\middle|\; \min\{[\![\varphi]\!](w), x\} \leq [\![\psi]\!](w)\right\} \\ &= \begin{cases} 1 & \text{if } [\![\varphi]\!](w) \leq [\![\psi]\!](w), \\ [\![\psi]\!](w) & \text{else.} \end{cases} \end{aligned}$$

The two alternative implications induce corresponding alternatives for our interpretation of $\Box_i$. In classical PML, one can view necessity as a generalized conjunction (think: universal quantification) of implications, like so:

$$[\![\Box_i\varphi]\!](w) = \bigwedge_{w' \in \mathcal{W}} \left(w' \in [\![i]\!](w) \to w' \in [\![\varphi]\!]\right). \tag{3.1}$$

[2]Our logic would be rather better behaved if we had chosen the so-called Gödel negation defined by

$$[\![\neg\varphi]\!](w) = \begin{cases} 1 & \text{if } [\![\varphi]\!](w) = 0; \\ 0 & \text{else.} \end{cases}$$

However, we do not think that the Gödel negation makes much sense for our PDL logic in Section 4. See footnote 4 for a fuller explanation.

The formula $w' \in [\![i]\!](w) \to w' \in [\![\varphi]\!]$ is not a formula in PML, however, and we are free to choose how to interpret implication here. This choice is independent of whether we have chosen to interpret $\to$ in PML as S- or R-implication. Our alternatives are presented in the box below[3].

S-necessity:

$$\begin{aligned}[\![\Box_i\varphi]\!](w) &= \textstyle\bigwedge_{w'\in\mathcal{W}}\big(w' \notin [\![i]\!](w) \vee w' \in [\![\varphi]\!]\big)\\ &= \min\left\{\max\{1-[\![i]\!](w)(w'), [\![\varphi]\!](w')\} \;\mid\; w' \in \mathcal{W}\right\}.\end{aligned}$$

R-necessity:

$$\begin{aligned}[\![\Box_i\varphi]\!](w) &= \textstyle\bigwedge_{w'\in\mathcal{W}} \sup\left\{x \in [0,1] \;\mid\; \min\{w' \in [\![i]\!](w), x\} \le [\![\varphi]\!](w)\right\}\\ &= \inf\left\{[\![\varphi]\!](w') \;\mid\; w' \in \mathcal{W} \text{ and } [\![\varphi]\!](w') < [\![i]\!](w)(w')\right\}.\end{aligned}$$

In [6], the authors investigated R-necessity (but not by that name). As far as we know, this is the first presentation of R-necessity and so we will choose this as our interpretation. The corresponding weak modal operator is defined by

$$\begin{aligned}[\![\Diamond_i\varphi]\!](w) &= [\![\neg\Box_i\neg\varphi]\!](w)\\ &= \sup\left\{[\![\varphi]\!](w') \;\mid\; w' \in \mathcal{W} \text{ and } 1-[\![i]\!](w)(w') < [\![\varphi]\!](w')\right\}.\end{aligned}$$

The two alternative interpretations of $\Box_i$ share a remarkable number of properties, which we will discuss below. We also compare the two interpretations with a concrete model in Example 3.1.

We do not offer a philosophical justification for choosing one interpretation over the other.

We summarize our fuzzy semantics for PML in Table 3.

$$\begin{aligned}[\![\top]\!](w) &= 1\\ [\![\neg\varphi]\!](w) &= (\mathcal{W} \setminus [\![\varphi]\!])(w)\\ [\![\varphi\wedge\psi]\!](w) &= ([\![\varphi]\!] \cap [\![\psi]\!])(w)\\ [\![\varphi\to\psi]\!](w) &= \begin{cases}1 & \text{if } [\![\varphi]\!](w) \le [\![\psi]\!](w),\\ [\![\psi]\!](w) & \text{else;}\end{cases}\\ [\![\Box_i\varphi]\!](w) &= \inf\left\{[\![\varphi]\!](w') \;\mid\; w' \in \mathcal{W} \text{ and } [\![\varphi]\!](w') < [\![i]\!](w)(w')\right\}\end{aligned}$$

TABLE 3. A fuzzy semantics for PML

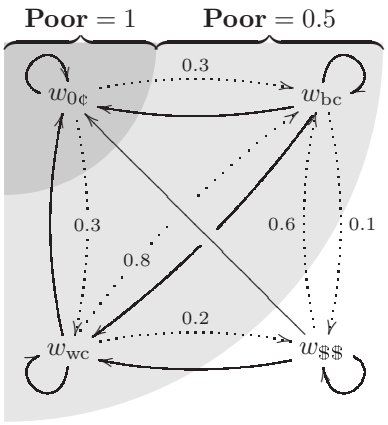
Note that the operations $\setminus$ and $\cap$ above are (standard) fuzzy operations. We call this the *fuzzy interpretation* of PML. It differs from some interpretations of PML, including [14].

Example 3.1. We will consider a simple language with a single atomic predicate **Poor** and a single conjugate pair of modal operators, $\Box$ and $\Diamond$. We denote the fuzzy accessibility relation by $\alpha : \mathcal{W} \to \mathcal{F}\mathcal{W}$. The predicate **Poor** represents the condition that a particular person, say Howard, has very little money.

Our model consists of four worlds. In world $w_{0¢}$, Howard is utterly penniless. In w_{bc}, he is a skilled blue collar worker, while in w_{wc}, he is a white collar worker earning comparable pay. In $w_{\$\$}$, he is a very wealthy man, having earned his fortune in business. Accordingly, the predicate

[3]Although we use the terms *S-* and *R-necessity*, we are not committed to any particular interpretation of the strong modal operator. We use "necessity" as a shorthand for "strong modal operator".

Poor evaluates to 1 in $w_{0¢}$, 0.5 in both w_{bc} and w_{wc} and 0 in $w_{\$\$}$. We illustrate this in Figure 2. Here, a transition $w \longrightarrow w'$ indicates that $\alpha(w)(w') = 1$ and a transition $w \overset{x}{\cdots\!\!\rightarrow} w'$ indicates $\alpha(w)(w') = x$. If no transition is shown from w to w', then $\alpha(w)(w') = 0$.



	□ def'n	$w_{0¢}$	w_{bc}	w_{wc}	$w_{\$\$}$
$[\![\mathbf{Poor}]\!]$	S	1	0.5	0.5	0
	R	1	0.5	0.5	0
$[\![\Box\mathbf{Poor}]\!]$	S	0.7	0.5	0.5	0
	R	1	0	0	0
$[\![\Box\Box\mathbf{Poor}]\!]$	S	0.7	0.5	0.5	0
	R	0	0	0	0
$[\![\Diamond\mathbf{Poor}]\!]$	S	1	1	1	1
	R	1	1	1	1
$[\![\neg\mathbf{Poor} \vee \Box\mathbf{Poor}]\!]$	S	0.7	0.5	0.5	1
	R	1	0.5	0.5	1
$[\![\neg\Box\mathbf{Poor} \vee \mathbf{Poor}]\!]$	S	1	0.5	0.5	1
	R	1	1	1	1
$[\![\mathbf{Poor} \to \Box\mathbf{Poor}]\!]$	S	0.7	1	1	1
	R	1	0	0	1
$[\![\Box\mathbf{Poor} \to \mathbf{Poor}]\!]$	S	1	1	1	1
	R	1	1	1	1

FIGURE 2. A fuzzy set PML model

In each world, downward mobility is considered a strong possibility, although for Howard the tycoon, the blue collar job seems somewhat less likely since it involves training he's never had. Upward mobility is considered a more remote possibility, and the transition structure reflects this. For Howard the penniless, the world in which he is wealthy is entirely irrelevant. The accessibility relation is reflexive.

We give the membership function for a handful of example formulas in Figure 2. This includes a comparison of R- and S-implications. In addition, we calculate the membership function for each formula twice, once for R-necessity and once for R-necessity. Not surprisingly, R-necessity tends to the extremes 0 and 1 more often than S-necessity.

For each $x \in [0,1]$, one may define a satisfaction relation $\models_x^{\mathbf{M}} \subseteq \mathcal{W} \times \Phi$ expressing that in a world w, the formula φ is confirmed to degree x. Explicitly, $w \models_x^{\mathbf{M}} \varphi$ iff $[\![\varphi]\!](w) \geq x$. There is a corresponding strong relation, replacing $\geq$ with $>$. We extend the definition in the usual ways:

- $\models_x^{\mathbf{M}} \varphi$ iff $w \models_x^{\mathbf{M}} \varphi$ for each $w \in \mathcal{W}$;
- $\models_x \varphi$ iff for every $\mathbf{M}$ we have $\models_x^{\mathbf{M}} \varphi$;
- $\models \varphi$ iff $\models_1 \varphi$; in this case, we say that φ is *valid* (for fuzzy PML).

We say that a rule of inference $\varphi_1, \ldots, \varphi_n / \psi$ is *(globally) sound* for our fuzzy set semantics if for each $x \in [0,1]$ and model $\mathbf{M}$, if $\models_x^{\mathbf{M}} \varphi_i$ for each $1 \leq i \leq n$, then $\models_x^{\mathbf{M}} \psi$. In other words, the above rule is sound iff

$$\min\{\inf\{[\![\varphi_i]\!](w) \mid w \in \mathcal{W}\} \mid 1 \leq i \leq n\} \leq \inf\{[\![\psi]\!](w) \mid w \in \mathcal{W}\} \tag{3.2}$$

If there is a derivation of $\varphi_1, \ldots, \varphi_n / \psi$ in which each inference is sound, then every model $\mathbf{M}$ satisfying each φ_i to degree x also satisfies ψ to degree x. In particular, if each φ_i is valid, then so is ψ.

For our presentation, it is useful to introduce a secondary notion. We say that a rule $\varphi_1, \ldots \varphi_n / \psi$ is *locally sound* iff for each model $\mathbf{M}$ and world w,

$$\min\{[\![\varphi_i]\!](w) \mid 1 \leq i \leq n\} \leq [\![\psi]\!](w).$$

Clearly, if a rule is locally sound, then it is sound.

de Morgan axioms

Involution	$\neg\neg\varphi \leftrightarrow \varphi$
Commutativity	$\varphi \wedge \psi \leftrightarrow \psi \wedge \varphi$
	$\varphi \vee \psi \leftrightarrow \psi \vee \varphi$
Associativity	$\varphi \wedge (\psi \wedge \vartheta) \leftrightarrow (\varphi \wedge \psi) \wedge \vartheta$
	$\varphi \vee (\psi \vee \vartheta) \leftrightarrow (\varphi \vee \psi) \vee \vartheta$
Distributivity	$\varphi \wedge (\psi \vee \vartheta) \leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \vartheta)$
	$\varphi \vee (\psi \wedge \vartheta) \leftrightarrow (\varphi \vee \psi) \wedge (\varphi \vee \vartheta)$
Idempotence	$\varphi \wedge \varphi \leftrightarrow \varphi$
	$\varphi \vee \varphi \leftrightarrow \varphi$
Absorption	$\varphi \wedge (\varphi \vee \psi) \leftrightarrow \varphi$
	$\varphi \vee (\varphi \wedge \psi) \leftrightarrow \varphi$
Absorption by $\bot$ and $\top$	$\varphi \wedge \bot \leftrightarrow \bot$
	$\varphi \vee \top \leftrightarrow \top$
Identity	$\varphi \wedge \top \leftrightarrow \varphi$
	$\varphi \vee \bot \leftrightarrow \varphi$
de Morgan's laws	$\neg(\varphi \wedge \psi) \leftrightarrow \neg\varphi \vee \neg\psi$
	$\neg(\varphi \vee \psi) \leftrightarrow \neg\varphi \wedge \neg\psi$

Implication axioms

Reflexivity	$\varphi \to \varphi$
Weakening	$(\varphi \to \psi) \to ((\varphi \wedge \vartheta) \to \psi)$
	$(\varphi \to \psi) \to (\varphi \to (\psi \vee \vartheta))$
Currying	$((\varphi \wedge \psi) \to \vartheta) \leftrightarrow (\varphi \to (\psi \to \vartheta))$
Preserves $\wedge$	$((\varphi \to \psi) \wedge (\varphi \to \vartheta)) \leftrightarrow (\varphi \to (\psi \wedge \vartheta))$
Preserves $\top$	$\varphi \to \top$

Inference rules

Modus Ponens	$\varphi, \varphi \to \psi / \psi$
Cut	$\varphi \to \psi, \psi \to \vartheta / \varphi \to \vartheta$
Necessitation	$\varphi / \Box_i \varphi$

TABLE 4. de Morgan algebra and implication axioms

The proof of the following proposition is straightforward. We omit it here.

Proposition 3.2. *Modus ponens and cut are locally sound.*

In the following proposition, we discuss the validity of some familiar axioms. We do not assert that this is a complete axiomatization. Our main interests are in providing a semantics for means-end relations rather than a complete deductive system and so we postpone close examination of the deductive calculus for later work.

For a discussion of completeness of de Morgan algebras with implications, see [12] and for a discussion of issues regarding complete proof systems for fuzzy logic, see [13].

Proposition 3.3. *The axioms summarized in Table 4 are valid for fuzzy PML. Furthermore, so is the K axiom*

$$\Box_i(\varphi \to \psi) \to (\Box_i\varphi \to \Box_i\psi).$$

Proof. The proofs for the axioms in Table 4 (which is taken from [11]) are easy and well known. We present the proof that K is valid.

The implication $\Box_i(\varphi \to \psi) \to (\Box_i\varphi \to \Box_i\psi)$ is valid just in case , in each model $\mathbf{M}$ and world w, either the antecedent evaluates as less likely than the consequent, i.e.

$$[\![\Box_i(\varphi \to \psi)]\!](w) \leq [\![\Box_i\varphi \to \Box_i\psi]\!](w),$$

or the consequent evaluates to 1, i.e. $[\![\Box_i\varphi \to \Box_i\psi]\!](w) = 1$. Now, the consequent evaluates to 1 just in case $[\![\Box_i\varphi]\!](w) \leq [\![\Box_i\psi]\!](w)$, and otherwise $[\![\Box_i\varphi \to \Box_i\psi]\!](w) = [\![\Box_i\psi]\!](w)$. Thus, we want to show that for every $\mathbf{M}$ and w, at least one of the inequalities below hold.

$$[\![\Box_i(\varphi \to \psi)]\!](w) \leq [\![\Box_i\psi]\!](w),$$
$$[\![\Box_i\varphi]\!](w) \leq [\![\Box_i\psi]\!](w).$$

Let x be the minimum of the two left hand sides.

We wish to show that x is less than or equal to $[\![\Box_i\psi]\!](w)$, i.e. that for each w' such that $[\![\psi]\!](w') < [\![i]\!](w)(w')$ we have $x \leq [\![\psi]\!](w')$. Let such w' be given. By the local soundness of modus ponens, we know that $[\![\varphi \to \psi]\!](w') \leq [\![\psi]\!](w')$ or $[\![\varphi]\!](w') \leq [\![\psi]\!](w')$. Let us suppose the former. Then

$$[\![\varphi \to \psi]\!](w') \leq [\![\psi]\!](w') < [\![i]\!](w)(w')$$

and so

$$[\![\varphi \to \psi]\!](w') \in \{[\![\varphi \to \psi]\!](v) \mid v \in \mathcal{W} \text{ and } [\![\varphi \to \psi]\!](v) < [\![i]\!](w)(v)\}. \tag{3.3}$$

Since $[\![\Box_i(\varphi \to \psi)]\!](w)$ is the infimum of the right hand side of (3.3), we see that

$$[\![\Box_i(\varphi \to \psi)]\!](w) \leq [\![\varphi \to \psi]\!](w'). \tag{3.4}$$

Consequently

$$\begin{aligned} x &\leq [\![\Box_i(\varphi \to \psi)]\!](w) && \text{(by definition of } x\text{)} \\ &\leq [\![\varphi \to \psi]\!](w') && \text{(by (3.4))} \\ &\leq [\![\psi]\!](w') && \text{(by case assumption).} \end{aligned}$$

The case $[\![\varphi]\!](w') \leq [\![\psi]\!](w')$ is similar. □

We turn our attention to necessitation, which is globally but not locally sound (just as in the classical case).

Proposition 3.4. *The rule of necessitation $\varphi/\Box_i\varphi$ is globally sound. That is,*

$$\inf\{[\![\varphi]\!](w) \mid w \in \mathcal{W}\} \leq \inf\{[\![\Box_i\varphi]\!](w) \mid w \in \mathcal{W}\}$$

Proof. Let x and $\mathbf{M}$ be given, satisfying that for every $w \in \mathcal{W}$, we have $[\![\varphi]\!](w) \geq x$. Then, for each $w \in \mathcal{W}$, we have

$$\begin{aligned} \inf\{[\![\Box_i\varphi]\!](w) \mid w \in \mathcal{W}\} &= \inf\{\inf\{[\![\varphi]\!](w') \mid w' \in \mathcal{W} \text{ and } [\![\varphi]\!](w') < [\![i]\!](w)(w')\} \mid w \in \mathcal{W}\} \\ &\geq \inf\{\inf\{[\![\varphi]\!](w') \mid w' \in \mathcal{W}\} \mid w \in \mathcal{W}\} \\ &= \inf\{[\![\varphi]\!](w) \mid w \in \mathcal{W}\}. \end{aligned}$$

□

Corollary 3.5. *The distributivity axiom $\Box_i(\varphi \wedge \psi) \to (\Box_i\varphi \wedge \Box_i\psi)$ is valid.*

Proof. This is an easy derivation via K, necessitation and modus ponens. □

We have presented proofs in terms of R-necessity, but in fact, all of the above theorems apply if we use the S-necessity discussed in [6]. Remarkably, the two interpretations are indistinguishable when we look at typical completeness results for other axioms, too. That is, the same classical results relating axioms to their classes of models are valid for the R-necessity iff they are valid for the S-necessity. See Table 5.

The results for the S-necessity were reported in [6] and [5]. Proofs for the corresponding claims (as well as definitions for the relevant properties) regarding the R-necessity are found in the appendix.

			Holds for...	
Name	Axiom	Classical result	R	S
D	$\Box_i\varphi \to \Diamond_i\varphi$	serial: $[\![i]\!](w) \neq \emptyset$	Y	Y
T	$\Box_i\varphi \to \varphi$	reflexive: $w \in [\![i]\!](w)$	Y	Y
B	$\varphi \to \Box_i\Diamond_i\varphi$	symmetric: $w' \in [\![i]\!](w) \to w \in [\![i]\!](w')$	N	N
4	$\Box_i\varphi \to \Box_i\Box_i\varphi$	transitive: $w' \in [\![i]\!](w) \wedge w'' \in [\![i]\!](w') \to w'' \in [\![i]\!](w)$	Y	Y
5	$\Diamond_i\varphi \to \Box_i\Diamond_i\varphi$	euclidean: $w', w'' \in [\![i]\!](w) \to w'' \in [\![i]\!](w')$	N	N

TABLE 5. Axioms and their corresponding frame conditions

4. Fuzzy set semantics for PDL

In the previous section, we discussed fuzzy semantics for propositional modal logic. A fuzzy model consisted of a set $\mathcal{W}$ of worlds, a fuzzy valuation $[\![-]\!] : \Phi_0 \to \mathcal{F}\mathcal{W}$ and an interpretation $[\![-]\!] : I \to (\mathcal{F}\mathcal{W})^{\mathcal{W}}$. In this section, we continue with the fuzzy valuation, but we will restrict the set of interpretations in order to reflect what we consider the distinctive features of dynamic logic with a fuzzy set interpretation. Also, for technical reasons, we cannot reasonably interpret non-deterministic choice $\alpha \cup \beta$ or iteration α^* in this setting, so we omit these constructions from our language until Section 5.

Our interpretation of the dynamic modal operators for PDL is based on the observation that, for each action α and world w, there is at most *one* world which "actually" results from doing α in w. In this respect, it differs from the accessibility relation for the modalities of PML, in which a (fuzzy) *set* of worlds is relevant in a particular world.

Let $\mathcal{D}X$ denote the set of *discrete subdistributions* on X, i.e.

$$\mathcal{D}X = \big\{p : X \to [0,1] \;\big|\; \sum_{x\in X} p(x) \leq 1\big\}.$$

Note that $\mathcal{D}$ is a sub-functor of the fuzzy subset functor $\mathcal{F}$. Also, notice that for crisp sets $S \subseteq X$, we have $S \in \mathcal{D}X$ iff S is a singleton. In keeping with our fuzzy set notations, we write $\{x\}$ for the distribution assigning 1 to x and 0 to all other elements of X.

A *fuzzy PDL frame* for Π_0 is a set $\mathcal{W}$ of worlds with an interpretation

$$[\![-]\!] : \Pi_0 \to (\mathcal{D}\mathcal{W})^{\mathcal{W}}$$

and a *fuzzy PDL model* extends a frame by adding a valuation $[\![-]\!] : \Phi_0 \to \mathcal{F}\mathcal{W}$. Given an atomic action m and worlds w and w', the value $[\![m]\!](w)(w')$ is a measure of the likelihood that doing m in w results in w'. The value $1 - \sum_{w'\in\mathcal{W}}[\![m]\!](w)(w')$ represents the likelihood that m cannot be performed at all, i.e. that m is *prohibited* in w. We denote this $[\![m]\!](w)(\emptyset)$.

In Section 3 we interpreted the fuzzy proposition $\Box_i\varphi$ in terms of a universally quantified implication. In particular, our definitions yielded $w \models_x \Box_i\varphi$ iff $w' \in [\![i]\!](w)$ implies $w' \in [\![\varphi]\!]$ (for some interpretation of "implies"). This universal requirement makes less sense in a dynamic context, since the actual outcome of α will be at most one world, not a set of worlds. Instead of defining $[\![[\alpha]\varphi]\!]$

in terms of universals, we use a notion of expected value, namely the mean of all the likelihoods involved:

$$[\![[\alpha]\varphi]\!](w) = [\![\alpha]\!](w)(\emptyset) + \sum_{w' \in \mathcal{W}} [\![\alpha]\!](w)(w') \cdot [\![\varphi]\!](w')$$

and, by conjugacy,

$$\begin{aligned}[\![\langle\alpha\rangle\varphi]\!](w) &= [\![\neg[\alpha]\neg\varphi]\!](w) \\ &= [\![[\alpha]\varphi]\!](w) - [\![\alpha]\!](w)(\emptyset).\end{aligned}$$

Thus, we calculate the dynamic operators in terms of the probabilities that the outcome of doing α will result in a world satisfying φ. The only difference between the two calculations is that in the strong dynamic operator, $[\![\alpha]\!](w)(\emptyset)$ is a summand. This is consistent with the usual convention that, if α is prohibited in w, then $w \models [\alpha]\varphi$, but it entails that $\langle\alpha\rangle\varphi \to [\alpha]\varphi$ is valid for our semantics.

If we are to interpret $\langle\alpha\rangle\varphi$ and $[\alpha]\varphi$ as fuzzy sets of worlds, then we should understand which vague propositions these formulas represent. We propose that $\langle\alpha\rangle\varphi$ represents "α reliably realizes φ," or "α is an effective means to φ," which we take to be roughly synonymous. Both of these are vague since they depend on "reliably" or "effective", both of which are fairly fuzzy terms. The value $[\![\langle\alpha\rangle\varphi]\!](w)$ is thus the degree to which, in w, it is the case that α is an effective means to φ. It is calculated[4] as the probability that the world which results from doing α in w satisfies φ.

The informal interpretation of $[\alpha]\varphi$ is a bit harder to motivate. The so-called strong operator gives a weaker means-end relation than the weak in our semantics, since

$$[\![[\alpha]\varphi]\!](w) = [\![\langle\alpha\rangle\varphi]\!](w) + [\![\langle\alpha\rangle\top]\!](w).$$

The formula $\neg\langle\alpha\rangle\top$ expresses the degree to which α can reliably realize $\top$, i.e. that α can reliably be performed. Thus, $[\alpha]\varphi$ can be interpreted as "α is an effective means to φ or α is (probably) prohibited"—but here, the disjunction is interpreted as the Łukasiewicz disjunction $\underline{\vee}$ defined by $x \underline{\vee} y = \min(x + y, 1)$, not the standard disjunction from Section 3. We abbreviate this vague proposition by saying α is a *weakly effective means* to φ.

We summarize the extensions of the valuation $[\![-]\!] : \Phi_0 \to \mathcal{F}\mathcal{W}$ and the interpretation $[\![-]\!] : \Pi_0 \to (\mathcal{D}\mathcal{W})^{\mathcal{W}}$ in Table 6.

Call the valuation *crisp* if for each atomic proposition P the set $[\![P]\!]$ is crisp and call the interpretation of Π_0 *crisp* if similarly each set $[\![m]\!](w)$ is crisp (and consequently either a singleton or empty). There is conceptual interest in focusing on crisp valuations, since one would like to separate issues of vagueness or other forms of fuzzy predication from the fuzziness that arises from uncertain outcomes. Nonetheless, we won't restrict to crisp valuations since it adds little to the logical properties of our theory.

When both the valuation and interpretation of actions are crisp, then the above definitions restrict to a special case of the PDL semantics from Section 2. Namely, it restricts to the case where each $[\![m]\!](w)$ is either empty or a singleton. This is a special kind of deterministic semantics, which is often represented by $[\![-]\!] : \Pi_0 \to (1 + \mathcal{W})^{\mathcal{W}}$.

Some of the semantics can be easily understood in terms of linear algebra. Let us simplify the presentation by assuming that $\mathcal{W}$ is finite, say, $\mathcal{W} = \{w_1, w_2, \ldots, w_n\}$. A particular subdistribution

[4]It is this interpretation of $\langle\alpha\rangle\varphi$ that persuaded us away from Gödel negation in favor of the standard negation. With Gödel negation, if the probability that α realizes φ in w is non-zero, then the truth value of the statement "α is not a reliable means to φ in w" is 0. That is, if there is any possibility that α yields a φ-world, then there is no truth to the statement that α is not a reliable means to φ.

This does not seem very sensible to us. It seems more reasonable to use the standard negation and resign ourselves to the undesirable logical properties than to accept such remarkably unintuitive interpretations regarding fairly basic vague predicates.

On formulas

$$[\![\top]\!](w) = 1$$

$$[\![\neg\varphi]\!](w) = (\mathcal{W} \setminus [\![\varphi]\!])(w)$$

$$[\![\varphi \wedge \psi]\!](w) = ([\![\varphi]\!] \cap [\![\psi]\!])(w)$$

$$[\![\varphi \to \psi]\!](w) = \begin{cases} 1 & \text{if } [\![\varphi]\!](w) \leq [\![\psi]\!](w), \\ [\![\psi]\!](w) & \text{else} \end{cases}$$

$$[\![[\alpha]\varphi]\!](w) = [\![\alpha]\!](w)(\emptyset) + \textstyle\sum_{w' \in \mathcal{W}} [\![\alpha]\!](w)(w') \cdot [\![\varphi]\!](w')$$

On actions

$$[\![\alpha;\beta]\!](w)(w') = \textstyle\sum_{w'' \in \mathcal{W}} [\![\alpha]\!](w)(w'') \cdot [\![\beta]\!](w'')(w')$$

$$[\![\varphi?]\!](w)(w') = \begin{cases} [\![\varphi]\!](w) & \text{if } w = w'; \\ 0 & \text{else.} \end{cases}$$

TABLE 6. A fuzzy semantics for PDL

$p \in \mathcal{D}(\mathcal{W})$ corresponds to a vector (i.e. $n \times 1$ matrix)

$$M_p = \begin{pmatrix} p(w_1) \\ p(w_2) \\ \vdots \\ p(w_n) \end{pmatrix},$$

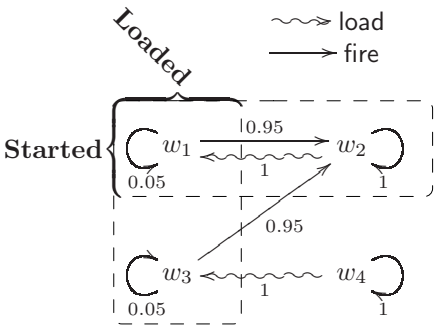
in particular, a vector whose elements sum to no more than one. Similarly, a function $f : \mathcal{W} \to \mathcal{D}\mathcal{W}$ corresponds to an $n \times n$ matrix M_f in which each row sums to no more than one. If we apply these observations to $[\![\varphi]\!]$ and $[\![\alpha]\!]$, respectively, we will observe that

$$M_{[\![\alpha]\!]} M_{[\![\varphi]\!]} = \begin{pmatrix} [\![\alpha]\!](w_1)(w_1) & [\![\alpha]\!](w_1)(w_2) & \cdots & [\![\alpha]\!](w_1)(w_n) \\ [\![\alpha]\!](w_2)(w_1) & [\![\alpha]\!](w_2)(w_2) & \cdots & [\![\alpha]\!](w_2)(w_n) \\ \vdots & \vdots & \ddots & \vdots \\ [\![\alpha]\!](w_n)(w_1) & [\![\alpha]\!](w_n)(w_2) & \cdots & [\![\alpha]\!](w_n)(w_n) \end{pmatrix} \begin{pmatrix} [\![\varphi]\!](w_1) \\ [\![\varphi]\!](w_2) \\ \vdots \\ [\![\varphi]\!](w_n) \end{pmatrix} = \begin{pmatrix} [\![\langle\alpha\rangle\varphi]\!](w_1) \\ [\![\langle\alpha\rangle\varphi]\!](w_2) \\ \vdots \\ [\![\langle\alpha\rangle\varphi]\!](w_n) \end{pmatrix}$$

In other words, $M_{[\![\langle\alpha\rangle]\!]\varphi} = M_{[\![\alpha]\!]} \cdot M_{[\![\varphi]\!]}$ (it is rather more natural to discuss $\langle\alpha\rangle$ than $[\alpha]$ in regards to the matrix operations). Similarly, $M_{[\![\alpha;\beta]\!]} = M_{[\![\alpha]\!]} M_{[\![\beta]\!]}$ (so composition is matrix multiplication) and

$$M_{[\![\varphi?]\!]} = \begin{pmatrix} [\![\varphi]\!](w_1) & 0 & \cdots & 0 \\ 0 & [\![\varphi]\!](w_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & [\![\varphi]\!](w_n) \end{pmatrix}$$

Example 4.1. We revisit the footrace from Example 2.1, exploiting the extra flexibility of the fuzzy set semantics. Previously, we included transitions $w_1 \to w_1$ and $w_3 \to w_3$ to represent the possibility that the gun misfires. But one might expect this occurrence to be less probable as a successful firing of the gun. We've assigned the probability of failure to ignite 0.05, the probability of successfully firing 0.95 and every other relevant probability is 1. We do not assume that a misfire increases the probability of subsequent misfires.



	w_1	w_2	w_3	w_4
$[\![\mathbf{Started}]\!]$	1	1	0	0
$[\![\mathbf{Loaded}]\!]$	0	1	0	1
$[\![[\mathsf{fire}]\mathbf{Started}]\!]$	1	1	0.95	0
$[\![\langle\mathsf{fire}\rangle\mathbf{Started}]\!]$	1	1	0.95	0
$[\![[\mathsf{load}]\mathbf{Loaded}]\!]$	1	1	1	1
$[\![\langle\mathsf{load}\rangle\mathbf{Loaded}]\!]$	0	1	0	1
$[\![[\mathbf{Loaded}?;\mathsf{fire}]\mathbf{Started}]\!]$	1	1	0.95	1
$[\![\langle\mathbf{Loaded}?;\mathsf{fire}\rangle\mathbf{Started}]\!]$	1	0	0.95	0
$[\![[\mathsf{load};\mathsf{fire}]\mathbf{Started}]\!]$	1	1	1	0.95
$[\![\langle\mathsf{load};\mathsf{fire}\rangle\mathbf{Started}]\!]$	0	1	0	0.95

FIGURE 3. A sample PDL model

In Figure 3, we present the fuzzy version of our footrace example, together with the interpretations of those formulas from Figure 1 (omitting those involving iteration and choice).

Reasoning in terms of expected outcomes is rather different than quantifying over all possible outcomes, as one does in classical PDL. Consequently, we lose some of the classical axioms for PDL, and also some of the rules of inference. First, we give the (modest) positive results and then discuss the negative. Naturally, we omit axioms regarding non-deterministic choice and iteration. We summarize the results in Table 7.

Positive results	**Negative results**
Axioms: De Morgan and Implication axioms (Table 4) are valid. Composition axiom (Table 2) is valid.	**Axioms:** K, Distributivity and Test (Table 1) are not valid.
Rules: Modus ponens, cut and necessitation are sound.	

TABLE 7. Summary of the logical features of fuzzy PDL

Proposition 4.2. *The fuzzy PDL semantics satisfies the axioms in Table 4 and also the composition axiom*

$$[\alpha;\beta]\varphi \leftrightarrow [\alpha][\beta]\varphi.$$

The inference rules modus ponens and cut are locally sound and necessitation globally sound.

Proof. We omit the proof of the de Morgan algebra equivalences and the composition axiom follows from straightforward calculation. The local soundness of modus ponens and cut goes as before. For

necessitation, let x and $\mathbf{M}$ be given and suppose that for every $w \in \mathcal{W}$, we have $[\![\varphi]\!](w) \geq x$.

$$\begin{aligned}
[\![[\alpha]\varphi]\!](w) &= [\![\alpha]\!](w)(\emptyset) + \sum_{w' \in \mathcal{W}} [\![\alpha]\!](w)(w') \cdot [\![\varphi]\!](w') \\
&\geq [\![\alpha]\!](w)(\emptyset) + \sum_{w' \in \mathcal{W}} [\![\alpha]\!](w)(w') \cdot x \\
&= 1 - \sum_{w' \in \mathcal{W}} [\![\alpha]\!](w)(w') + \sum_{w' \in \mathcal{W}} [\![\alpha]\!](w)(w') \cdot x \\
&= 1 + (x-1) \cdot \sum_{w' \in \mathcal{W}} [\![\alpha]\!](w)(w') \\
&\geq 1 + (x-1) = x. \qquad (*)
\end{aligned}$$

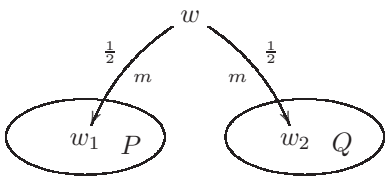
Inequality (*) follows from the fact that $x - 1 \leq 0$ and $\sum_{w' \in \mathcal{W}} [\![\alpha]\!](w)(w') \leq 1$. □

Now for the negative features. We start with the simplest observation: the Test axiom is not valid, i.e. it is not the case that $\models [\psi?]\varphi \leftrightarrow (\psi \to \varphi)$. Applying our interpretation of $[\psi?]$, we see that

$$[\![[\psi?]\varphi]\!](w) = [\![\neg\psi]\!](w) + [\![\psi]\!](w) \cdot [\![\varphi]\!](w).$$

This is clearly not the same as either interpretation of $\psi \to \varphi$ given in the box on page 6. As long as we interpret the dynamic operators in terms of expected outcomes (i.e. weighted averages), we cannot expect the conditional operator to act as implication (though when we restrict to crisp dynamic interpretation and valuation, we do satisfy the axiom as desired).

The fuzzy PDL semantics also does not satisfy the distributivity equivalence $[\alpha](\varphi \wedge \psi) \leftrightarrow ([\alpha]\varphi \wedge [\alpha]\psi)$. For instance, consider the following situation, in which m has a probability $\frac{1}{2}$ of yielding w_1 in w, and also a probability $\frac{1}{2}$ of yielding w_2.



In this case $[\![[m](P \wedge Q)]\!](w) = 0$ but $[\![[m]P \wedge [m]Q]\!](w) = \frac{1}{2}$.

Since the distributivity axiom $[\alpha](\varphi \wedge \psi) \leftrightarrow ([\alpha]\varphi \wedge [\alpha]\psi)$ is derivable from K using the axioms and rules of inference in Table 4, it follows that K is not valid for our fuzzy PDL semantics. But the failure to accommodate distributivity is in keeping with the usual motivations of fuzzy set theory. Let us consider the conjunction $\langle\alpha\rangle\varphi \wedge \langle\alpha\rangle\psi$. This asserts two local means-end relations: that α is a reliable means to φ and that α is a reliable means to ψ. The fuzzy set semantics gives a measure of the degree to which each conjunct is true, and thereby the degree to which the conjunction is true. But this is different than the degree to which α is a reliable means to $\varphi \wedge \psi$, which is our interpretation of $\langle\alpha\rangle(\varphi \wedge \psi)$. In other words, the distributive axiom simply doesn't reflect the intended interpretation of our fuzzy set semantics and so its failure is a feature rather than a shortcoming of our interpretation.

In the next section, we introduce more complex models that allow for a representation of non-deterministic choice and iteration.

5. Minimal fuzzy models

Non-determinism is a natural feature of PDL, but it is missing from our fuzzy models from the previous section. Probabilities are useful for introducing efficacy, but by replacing the powerset operator $\mathcal{P}$ by the subdistribution operator $\mathcal{D}$, we lost some of the expressiveness of PDL. In particular, there is no natural way to interpret non-deterministic choice using the models from Section 4. We don't want to assign probabilities to the outcome of a non-deterministic choice, since this would

be a fundamental re-interpretation—it would replace the non-deterministic feature with a related probabilistic choice. But probabilistic choices aren't non-deterministic in the usual sense.

We fix this problem by turning to a kind of neighborhood semantics. A *minimal fuzzy PDL frame* for Π_0 is a set $\mathcal{W}$ of worlds, together with a dynamic interpretation

$$[\![-]\!] : \Pi_0 \to (\mathcal{PDW})^{\mathcal{W}}$$

. This interpretation takes an atomic action m and world w to a *set* of discrete subdistributions $[\![m]\!](w)$ on $\mathcal{W}$. The elements of $[\![m]\!](w)$ represent alternatives which may be selected non-deterministically. A *minimal fuzzy PDL model* adds a valuation $[\![-]\!] : \Phi_0 \to \mathcal{FW}$ to a minimal fuzzy PDL frame.

Our dynamic structure mixes fuzzy and non-deterministic features. Thus, we define our semantics by mixing the non-deterministic semantics of Section 2 with the fuzzy set semantics of the previous section.

For technical reasons, we require that our set $\mathcal{W}$ of worlds is finite, so that all of the sums that appear here are finite sums and hence commute with limits. In particular, this means that for every family $(S_w)_{w\in\mathcal{W}}$ of sets, we have

$$\sum_{w\in\mathcal{W}} \inf S_w = \inf\{ \sum_{w\in\mathcal{W}} \sigma_w \mid \sigma \in \prod_{w\in\mathcal{W}} S_w\}. \tag{5.1}$$

Here, σ is a $\mathcal{W}$-indexed vector $\sigma = \langle \sigma_{w_1}, \sigma_{w_2}, \ldots, \sigma_{w_n}\rangle$ and σ_w is the w'th projection of σ. We will be particularly interested in the product $\prod_{w\in\mathcal{W}}[\![\alpha]\!](w)$. In this case, elements σ of this product are vectors of subdistributions, so that for each w, $\sigma_w \in [\![\alpha]\!](w) \subseteq \mathcal{D}(\mathcal{W})$.

In the sequel, we use $p, q, \ldots$ to range over subdistributions on $\mathcal{W}$ and write $p(\emptyset)$ for $1 - \sum_{w\in\mathcal{W}} p(w)$. At a given world w, the interpretation $[\![\alpha]\!](w)$ of α in w is a set of subdistributions. We interpret $p \in [\![\alpha]\!](w)$ as follows: $p(w')$ as the probability that doing α in w yields w' *provided* that p is the subdistribution non-deterministically chosen. This serves as the motivation for our minimal fuzzy semantics presented in Table 8.

The formula $\langle\alpha\rangle\varphi$ is intended as "one *can* reliably realize φ by doing α," where "can" here is the can of ability described by Brown [2]. Put differently, $\langle\alpha\rangle\varphi$ means, "There is a manner of doing α to reliably realize φ." The conjugate formula $[\alpha]\varphi$ represents the vague proposition, "However α is done, it is a weakly effective means to φ."

We use the standard fuzzy infinitary disjunction (conjunction, resp.) to assign the least upper bound (greatest lower bound, resp.) of the relevant expected values in interpreting $\langle\alpha\rangle\varphi$ ($[\alpha]\psi$, resp.):

$$[\![\langle\alpha\rangle\varphi]\!](w) = \sup\left\{ \sum_{w'\in\mathcal{W}} p(w')\cdot[\![\varphi]\!](w') \;\middle|\; p \in [\![\alpha]\!](w)\right\}$$

$$[\![[\alpha]\varphi]\!](w) = \inf\left\{ p(\emptyset) + \sum_{w'\in\mathcal{W}} p(w')\cdot[\![\varphi]\!](w') \;\middle|\; p \in [\![\alpha]\!](w)\right\}$$

Composition looks complicated but the idea is straightforward: create a set consisting of every relevant composition of subdistributions. As in Section 4, we can express this in terms of linear algebra. Let M_p be the $n \times 1$ matrix representing the subdistribution p and let M_σ be the $n \times n$ matrix representing the vector σ, namely

$$M_\sigma = \begin{pmatrix} \sigma_{w_1}(w_1) & \sigma_{w_2}(w_1) & \cdots & \sigma_{w_n}(w_1) \\ \sigma_{w_1}(w_2) & \sigma_{w_2}(w_2) & \cdots & \sigma_{w_n}(w_2) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{w_1}(w_n) & \sigma_{w_2}(w_n) & \cdots & \sigma_{w_n}(w_n) \end{pmatrix}.$$

In the case that $\sigma \in \prod_{v\in\mathcal{W}}[\![\beta]\!](v)$, it represents one possible β-transition system. The subdistribution σ_{w_i} is the fuzzy set of possible outcomes of that transition in w_i. Composition, then,

On formulas

$$[\![\top]\!](w) = 1$$

$$[\![\neg\varphi]\!](w) = (\mathcal{W} \setminus [\![\varphi]\!])(w)$$

$$[\![\varphi \wedge \psi]\!](w) = ([\![\varphi]\!] \cap [\![\psi]\!])(w)$$

$$[\![\varphi \to \psi]\!](w) = \begin{cases} 1 & \text{if } [\![\varphi]\!](w) \leq [\![\psi]\!](w), \\ [\![\psi]\!](w) & \text{else} \end{cases}$$

$$[\![[\alpha]\varphi]\!](w) = \inf\left\{p(\emptyset) + \textstyle\sum_{w' \in \mathcal{W}} p(w') \cdot [\![\varphi]\!](w') \;\mid\; p \in [\![\alpha]\!](w)\right\}$$

On actions

$$[\![\alpha;\beta]\!](w) = \left\{\lambda w' \,.\, \textstyle\sum_{w'' \in \mathcal{W}} p(w'') \cdot \sigma_{w''}(w') \;\mid\; p \in [\![\alpha]\!](w),\ \sigma \in \prod_{v \in \mathcal{W}} [\![\beta]\!](v)\right\}$$

$$[\![\alpha \cup \beta]\!](w) = ([\![\alpha]\!] \cup [\![\beta]\!])(w)$$

$$[\![\alpha^*]\!](w) = \textstyle\bigcup_{n=0}^{\infty} [\![\underbrace{\alpha;\ldots;\alpha}_{n \text{ times}}]\!](w)$$

$$[\![\varphi?]\!](w) = \{p_\varphi^w\} \text{ where } p_\varphi^w(w') = \begin{cases} [\![\varphi]\!](w) & \text{if } w = w'; \\ 0 & \text{else.} \end{cases}$$

TABLE 8. A minimal fuzzy semantics for PDL

is given by

$$[\![\alpha;\beta]\!] = \left\{ M_\sigma M_p \;\middle|\; p \in [\![\alpha]\!](w),\ \sigma \in \prod_{v \in \mathcal{W}} [\![\beta]\!](v) \right\}.$$

Non-deterministic choice and iteration now can be defined exactly as in the standard PDL semantics, and the conditional operator is interpreted by a singleton consisting of its interpretation from Section 4. For iteration, we apply the convention that $[\![\alpha^0]\!](w) = \{w\}$ (in other words, M_{α^0} is the identity matrix), so that $\{w\} \in [\![\alpha^*]\!](w)$.

Example 5.1. We illustrate the minimal fuzzy model semantics by returning to the footrace example from the previous section. Previously, we had to defer on examining formulas involving iteration and choice, since the normal fuzzy set semantics has no natural definition for these constructions. Minimal fuzzy models correct this deficiency and allow us to complete the table from Figure 3, as shown in Table 9.

	w_1	w_2	w_3	w_4
$[\![[(\mathsf{load} \cup \mathsf{fire})^*]\mathbf{Started}]\!]$	1	1	0	0
$[\![\langle(\mathsf{load} \cup \mathsf{fire})^*\rangle\mathbf{Started}]\!]$	1	1	1	1

TABLE 9. The completion of the table in Figure 3.

Let us explicitly calculate the new entries in Table 9. First, we remind ourselves of the relevant definition.

$$[\![[(\mathsf{load} \cup \mathsf{fire})^*]\mathbf{Started}]\!](w) = \inf\{1 + \sum_{w' \in \mathcal{W}} [\![p]\!](w') \cdot ([\![\mathbf{Started}]\!](w') - 1) \mid p \in [\![(\mathsf{load} \cup \mathsf{fire})^*]\!](w)\}.$$

It is easy to see that, for all $p \in [\![(\mathsf{load} \cup \mathsf{fire})^*]\!](w_1)$ and $w \in \mathcal{W}$,

$$[\![p]\!](w) \cdot ([\![\mathbf{Started}]\!](w) - 1) = 0.$$

Thus,

$$[\![[(\mathsf{load} \cup \mathsf{fire})^*]\mathbf{Started}]\!](w_1) = 1.$$

The same reasoning applies to w_2. Once the race is started, no action stops it.

For w_3 and w_4, the reasoning is even simpler. By definition of iteration, the crisp singleton $\{w_3\}$ ($\{w_4\}$, resp.) is an element of $[\![(\mathsf{load} \cup \mathsf{fire})^*]\!](w_3)$ ($[\![(\mathsf{load} \cup \mathsf{fire})^*]\!](w_4)$, resp.), since it is the interpretation of the do-nothing action $(\mathsf{load} \cup \mathsf{fire})^0$. Thus

$$\begin{aligned}[\![[(\mathsf{load} \cup \mathsf{fire})^*]\mathbf{Started}]\!](w_3) &\leq 1 + \sum_{w' \in \mathcal{W}} \{w_3\}(w') \cdot ([\![\mathbf{Started}]\!](w') - 1) \\ &= 1 + 1 \cdot ([\![\mathbf{Started}]\!](w_3) - 1) \\ &= 0\end{aligned}$$

The same reasoning applies for w_4.

The reasoning for the weak dynamic operator is subtler and more interesting. Recall

$$[\![\langle(\mathsf{load} \cup \mathsf{fire})^*\rangle\mathbf{Started}]\!](w) = \sup\left\{ \sum_{w' \in \mathcal{W}} [\![p]\!](w') \cdot [\![\mathbf{Started}]\!](w') \;\middle|\; p \in [\![(\mathsf{load} \cup \mathsf{fire})^*]\!](w) \right\}.$$

The do-nothing action $(\mathsf{load} \cup \mathsf{fire})^0$ suffices to show that our formula evaluates to 1 in w_1 and w_2. For w_3, we note that

$$[\![\underbrace{\mathsf{fire}; \ldots; \mathsf{fire}}_{n \text{ times}}]\!](w_3) = \begin{cases} w_1 & \mapsto 0 \\ w_2 & \mapsto 1 - 0.05^n \\ w_3 & \mapsto 0.05^n \\ w_4 & \mapsto 0 \end{cases}$$

Thus

$$\begin{aligned}[\![\langle(\mathsf{load} \cup \mathsf{fire})^*\rangle\mathbf{Started}]\!](w_3) &\geq \sup\left\{ \sum_{w' \in \mathcal{W}} [\![\underbrace{\mathsf{fire}; \ldots; \mathsf{fire}}_{n \text{ times}}]\!](w_3)(w') \cdot [\![\mathbf{Started}]\!](w') \;\middle|\; n \in \mathbb{N} \right\} \\ &= \sup\{1 - 0.05^n \mid n \in \mathbb{N}\} \\ &= 1.\end{aligned}$$

The same reasoning applied to actions of the form $\mathsf{load}; \mathsf{fire}; \ldots; \mathsf{fire}$ yields the same result for w_4.

This completes our calculations for Table 9. Curiously, the two featured formulas evaluate to the same crisp sets as they did in Example 2.1. Again, we have $[\![[(\mathsf{load} \cup \mathsf{fire})^*]\mathbf{Started}]\!] = [\![\mathbf{Started}]\!]$ and $[\![\langle(\mathsf{load} \cup \mathsf{fire})^*\rangle\mathbf{Started}]\!] = \mathcal{W}$. But this is an accident of our example and not a symptom of a deeper fact.

We close this section with an examination of the logical features of minimal fuzzy PDL models, the good and the bad. These features are summarized in the table below.

Positive results	**Negative results**
Axioms: De Morgan, Implication and Composition axioms are valid. Choice and iteration (Table 2) are also valid.	**Axioms:** K, Distributivity and Test are not valid. Induction (Table 2) is not valid.
Rules: Modus ponens, cut and necessitation are sound.	

TABLE 10. Summary of the logical features of minimal fuzzy PDL

The following is the analogue for the minimal model PDL semantics of Proposition 4.2 (which relates to the fuzzy PDL semantics).

Proposition 5.2. *The minimal model PDL semantics satisfies the de Morgan and Implication axioms in Table 4 and also the composition axiom*

$$[\alpha;\beta]\varphi \leftrightarrow [\alpha][\beta]\varphi.$$

The inference rules modus ponens and cut are locally sound and necessitation globally sound.

Proof. The axioms, modus ponens and cut proofs are as in the proof of Proposition 4.2. We turn to the validity of the composition axiom.

By definition, we have

$$[\![[\alpha][\beta]\varphi]\!](w) = \inf\left\{1 + \sum_{w''\in\mathcal{W}} p(w'')\cdot(-1+\inf Q_{w''}) \;\middle|\; p \in [\![\alpha]\!](w)\right\},$$

where $Q_{w''} = \left\{1 + \sum_{w'\in\mathcal{W}} q(w')\cdot(-1+[\![\varphi]\!](w')) \;\middle|\; q\in[\![\beta]\!](w'')\right\}$. The term $(-1+\inf Q_{w''})$ simplifies to

$$\inf\left\{\sum_{w'\in\mathcal{W}} q(w')\cdot(-1+[\![\varphi]\!](w')) \;\middle|\; q\in[\![\beta]\!](w'')\right\}.$$

Thus, $[\![[\alpha][\beta]\varphi]\!](w)$ can be written

$$\inf\left\{1+\sum_{w''\in\mathcal{W}} p(w'')\cdot\inf\left\{\sum_{w'\in\mathcal{W}} q(w')\cdot(-1+[\![\varphi]\!](w')) \;\middle|\; q\in[\![\beta]\!](w'')\right\} \;\middle|\; p\in[\![\alpha]\!](w)\right\}.$$

We use the identity $a\cdot\inf\{x_i \mid i\in I\} = \inf\{a\cdot x_i \mid i\in I\}$ and the fact that infima commute with finite sums (Equation (5.1)) to rewrite this as

$$\inf\left\{1+\inf\left\{\sum_{w''\in\mathcal{W}} p(w'')\cdot\sum_{w'\in\mathcal{W}}\sigma_{w''}(w')\cdot(-1+[\![\varphi]\!](w')) \;\middle|\; \sigma\in\prod_{v\in\mathcal{W}}[\![\beta]\!](v)\right\} \;\middle|\; p\in[\![\alpha]\!](w)\right\}.$$

This in turn rewrites as

$$\inf\left\{1+\sum_{w''\in\mathcal{W}} p(w'')\cdot\sum_{w'\in\mathcal{W}}\sigma_{w''}(w')\cdot(-1+[\![\varphi]\!](w')) \;\middle|\; \sigma\in\prod_{v\in\mathcal{W}}[\![\beta]\!](v), p\in[\![\alpha]\!](w)\right\}.$$

Now we rewrite the term $\sum_{w''\in\mathcal{W}} p(w'')\cdot\sum_{w'\in\mathcal{W}}\sigma_{w''}(w')\cdot(-1+[\![\varphi]\!](w'))$ as

$$\sum_{w'\in\mathcal{W}}\left(\sum_{w''\in\mathcal{W}} p(w'')\cdot\sigma_{w''}(w')\right)\cdot(-1+[\![\varphi]\!](w')).$$

Here, σ and p range over the same values as in the definition of composition, and thus, we have

$$[\![[\alpha][\beta]\varphi]\!](w) = \inf\left\{1+\sum_{w'\in\mathcal{W}} r(w')\cdot(-1+[\![\varphi]\!](w')) \;\middle|\; r\in[\![\alpha;\beta]\!](w)\right\}.$$

The right hand side is exactly $[\![[\alpha;\beta]]\!]\varphi(w)$.

We conclude with the proof that necessitation is sound. Suppose that for all $w\in\mathcal{W}$ we have $[\![\varphi]\!](w)\geq x$. Then

$$\begin{aligned}
[\![[\alpha]\varphi]\!](w) &= \inf\left\{1+\sum_{w'\in\mathcal{W}} p(w')\cdot(-1+[\![\varphi]\!](w')) \;\middle|\; p\in[\![\alpha]\!](w)\right\}\\
&\geq \inf\left\{1+\sum_{w'\in\mathcal{W}} p(w')\cdot(-1+x) \;\middle|\; p\in[\![\alpha]\!](w)\right\}\\
&\geq \inf\{1+1\cdot(-1+x) \mid p\in[\![\alpha]\!](w)\} && \text{(because } x\leq 1\text{)}\\
&= x
\end{aligned}$$

□

Since the minimal fuzzy semantics reduces (in special cases) to the fuzzy semantics from the previous section, the negative results discussed there apply here too. Namely, our minimal fuzzy semantics does not satisfy the axioms Test, Distributivity or K.

Of course, our motivation in this section isn't to fix the problems of Section 4. We introduced the more complicated semantics so that we could represent non-deterministic choice and iteration. We close the section with a discussion of the relevant axioms for these constructions.

Proposition 5.3. *The Choice axiom* $[\alpha \cup \beta]\varphi \leftrightarrow ([\alpha]\varphi \wedge [\beta]\varphi)$ *and Iteration axiom* $[\alpha^*]\varphi \leftrightarrow (\varphi \wedge [\alpha][\alpha^*]\varphi)$ *are valid for the minimal fuzzy semantics.*

Sketch. The proof is a bit tedious, but straightforward. We sketch it here.

For the Choice axiom, the key step involves the identity

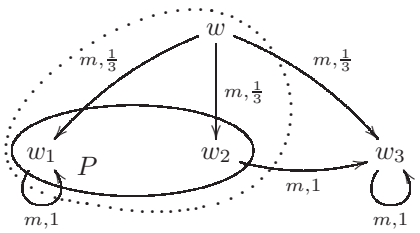
$$\inf(S \cup T) = \min\{\inf S, \inf T\}.$$

To confirm the Iteration axiom, one first shows that, for every model $\mathbf{M}$ and world w,

$$[\![\alpha; \alpha^*]\!](w) = \bigcup_{n=1}^{\infty} [\![\underbrace{\alpha; \ldots; \alpha}_{n \text{ times}}]\!](w).$$

This confirmation is routine and, with this identity, the Iteration axiom follows by another simple calculation. □

Our semantics fails to satisfy the Induction axiom. This is not too surprising, given that K is not a valid axiom. The model in Figure 4 is a counterexample.



	w	w_1	w_2	w_3
$[\![P]\!]$	$\frac{2}{3}$	1	1	0
$[\![[m]P]\!]$	$\frac{2}{3}$	1	0	0
$[\![P \to [m]P]\!]$	1	1	0	1

FIGURE 4. A counterexample to Induction

The region interior to the dotted curve marks the area where $[\![P]\!]$ evaluates to at least $\frac{2}{3}$ (in particular, $[\![P]\!](w) = \frac{2}{3}$), while in the interior of the solid curve $[\![P]\!]$ evaluates to 1 and $[\![P]\!](w_3) = 0$. We will calculate

$$[\![[m^*](P \to [m]P) \to (P \to [m^*]P)]\!](w).$$

First, one must calculate the set $[\![m^*]\!](w)$, by calculating $[\![m; \ldots; m]\!](w)$. For a single m, we have

$$[\![m]\!](w) = \begin{cases} w_1 & \mapsto \frac{1}{3} \\ w_2 & \mapsto \frac{1}{3} \\ w_3 & \mapsto \frac{1}{3} \end{cases}$$

For $n \geq 2$, we have

$$[\![\underbrace{m; \ldots; m}_{n \text{ times}}]\!](w) = \begin{cases} w_1 & \mapsto \frac{1}{3} \\ w_3 & \mapsto \frac{2}{3} \end{cases}$$

Name	Axiom	Frame Condition	Equivalent?
D	$\Box_i\varphi \to \Diamond_i\varphi$	serial: $\exists w' \,.\, [\![i]\!](w)(w') = 1$	Y
T	$\Box_i\varphi \to \varphi$	reflexive: $[\![i]\!](w)(w) = 1$	Y
B	$\varphi \to \Box_i\Diamond_i\varphi$	symmetric: $[\![i]\!](w)(w') = [\![i]\!](w')(w)$	N
4	$\Box_i\varphi \to \Box_i\Box_i\varphi$	transitive: $\min\{[\![i]\!](w)(w'), [\![i]\!](w')(w'')\} \leq [\![i]\!](w)(w'')$	Y
5	$\Diamond_i\varphi \to \Box_i\Diamond_i\varphi$	euclidean: $\min\{[\![i]\!](w)(w'), [\![i]\!](w)(w'')\} \leq [\![i]\!](w')(w'')$	N

TABLE 11. The axiom/frame condition correspondence for fuzzy PML

Consequently, $[\![m^*]\!](w) = \{\{w\}, [\![m]\!](w), [\![m;m]\!](w)\}$. We calculate

$$\begin{aligned}
[\![[m^*]P]\!](w) &= \inf\{[\![P]\!](w),\ \frac{1}{3}[\![P]\!](w_1) + \frac{1}{3}[\![P]\!](w_2) + \frac{1}{3}[\![P]\!](w_3),\ \frac{1}{3}[\![P]\!](w_1) + \frac{2}{3}[\![P]\!](w_3)\} \\
&= \inf\{\frac{2}{3},\ \frac{1}{3} + \frac{1}{3} + 0,\ \frac{1}{3} + 0\} \\
&= \frac{1}{3}.
\end{aligned}$$

$$\begin{aligned}
[\![[m^*](P \to [m]P)]\!](w) = \inf\{&[\![P \to [m]P]\!](w), \\
&\frac{1}{3}[\![P \to [m]P]\!](w_1) + \frac{1}{3}[\![P \to [m]P]\!](w_2) + \frac{1}{3}[\![P \to [m]P]\!](w_3), \\
&\frac{1}{3}[\![P \to [m]P]\!](w_1) + \frac{2}{3}[\![P \to [m]P]\!](w_3)\} \\
= \inf\{&1,\ \frac{1}{3} + 0 + \frac{1}{3},\ \frac{1}{3} + \frac{2}{3}\} \\
= \frac{2}{3}.&
\end{aligned}$$

Since $[\![P]\!](w) = \frac{2}{3} \geq [\![[m^*]P]\!](w)$, we see that $[\![P \to [m^*]P]\!](w) = [\![[m^*]P]\!](w) = \frac{1}{3}$ and hence

$$[\![[m^*](P \to [m]P) \to (P \to [m^*]P)]\!](w) = \frac{1}{3}.$$

APPENDIX A. PROPERTIES OF KRIPKE MODELS AND FUZZY INTERPRETATIONS

In Section 3, we presented a table showing the usual classical relationship between familiar axioms of modal logic and properties of the corresponding models. Some of these relations apply in fuzzy modal logic and others do not. Surprisingly, exactly the same relations hold regardless of whether we choose the R- or S-necessity.

In [5], Gounder presented the results for S-necessity. In this section, we present the corresponding results for R-necessity. We do not see any direct derivation from the theorems of [6], or a common proof of both.

Of course, in order to check the correspondence between various axioms and frame conditions, we must interpret the frame conditions in our fuzzy setting. We take the classical statement of the correspondences from [3] and give our interpretations in Table 11.

Before proceeding, we must also make clear what the claimed equivalence means. For this, we need to introduce a notion of *frame validity* which generalizes the notion of *model validity* from Section 3.

Definition 1. Fix an index set I and a non-empty set Φ_0 of atomic propositions. A *fuzzy frame* for I is a pair consisting of a set $\mathcal{W}$ of worlds and a fuzzy interpretation $[\![-]\!] : I \to (\mathcal{F}\mathcal{W})^{\mathcal{W}}$. We say that a formula φ is *valid* for $\langle \mathcal{W}, [\![-]\!] : I \to (\mathcal{F}\mathcal{W})^{\mathcal{W}} \rangle$ just in case for every valuation $[\![-]\!] : \Phi_0 \to \mathcal{F}\mathcal{W}$, the formula φ is valid for the model $\langle \mathcal{W}, [\![-]\!] : I \to (\mathcal{F}\mathcal{W})^{\mathcal{W}}, [\![-]\!] : \Phi_0 \to \mathcal{F}\mathcal{W} \rangle$.

In other words, a frame is a model without the valuation, and φ is valid for the frame F iff it's valid for every model built by adding a valuation to F. The equivalences in Table 11 can then be stated: The relevant axiom is valid for a frame F iff F satisfies the corresponding frame condition.

We turn now to proving the equivalences which hold and giving counterexamples for the others.

Proposition A.1. *The axiom D is valid for a frame F iff F is serial.*

Proof. First, we assume that F is serial and prove that D is valid. Accordingly, let the valuation $[\![-]\!] : \Phi_0 \to \mathcal{F}\mathcal{W}$ be given.

Let $w \in \mathcal{W}$ and we will show that $[\![\Box_i\varphi]\!](w) \leq [\![\Diamond_i\varphi]\!](w)$. Let w' be given such that $[\![i]\!](w)(w') = 1$, as in the hypothesis that F is serial.

Because $[\![i]\!](w)(w') = 1$, either $[\![\varphi]\!](w') < [\![i]\!](w)(w')$ or $[\![\varphi]\!](w') = 1$. Consequently,

$$[\![\Box_i\varphi]\!](w) = \inf\{[\![\varphi]\!](v) \mid v \in \mathcal{W} \text{ and } [\![\varphi]\!](v) < [\![i]\!](w)(v)\} \leq [\![\varphi]\!](w').$$

Also, by similar reasoning,

$$\begin{aligned}[\![\Diamond_i\varphi]\!](w) &= 1 - \inf\{1 - [\![\varphi]\!](v) \mid v \in \mathcal{W} \text{ and } 1 - [\![\varphi]\!](v) < [\![i]\!](w)(v)\} \\ &= \sup\{[\![\varphi]\!](v) \mid v \in \mathcal{W} \text{ and } 1 - [\![i]\!](w)(v) < [\![\varphi]\!](v)\} \\ &\geq [\![\varphi]\!](w')\end{aligned}$$

and thus $[\![\Box_i\varphi]\!](w) \leq [\![\varphi]\!](w') \leq [\![\Diamond_i\varphi]\!](w)$ and hence $[\![\Box_i\varphi \to \Diamond_i\varphi]\!](w) = 1$.

Now we prove the other direction. We will assume that F is not serial and show that D is not valid for F. Thus, we assume that there is a $w \in \mathcal{W}$ such that for all $w' \in \mathcal{W}$, $[\![i]\!](w)(w') < 1$. Pick $P \in \Phi_0$ and choose a valuation such that for all $v \in \mathcal{W}$,

$$[\![P]\!](v) = \sup\{[\![i]\!](w)(w') \mid w' \in \mathcal{W}\}.$$

We calculate

$$\begin{aligned}[\![\Box_i P]\!](w) &= \inf\{[\![P]\!](w') \mid w' \in \mathcal{W} \text{ and } [\![P]\!](w') < [\![i]\!](w)(w')\} \\ &= \inf\emptyset \\ &= 1 \\ [\![\Diamond_i P]\!](w) &= \sup\{[\![P]\!](w') \mid w' \in \mathcal{W} \text{ and } 1 - [\![i]\!](w)(w') < [\![P]\!](w')\} \\ &\leq \sup\{[\![i]\!](w)(w') \mid w' \in \mathcal{W}\} \\ &< 1\end{aligned}$$

Thus, $[\![\Box_i P \to \Diamond_i P]\!] = [\![\Diamond_i P]\!] < 1$ and so D is not valid for F. □

Proposition A.2. *The axiom T is valid for a frame F iff F is reflexive.*

Proof. Suppose that F is reflexive, i.e. that for every w, $[\![i]\!](w)(w) = 1$. By definition of $\to$, we have $[\![\Box_i\varphi \to \varphi]\!](w) \geq [\![\varphi]\!](w)$, so if $[\![\varphi]\!](w) \geq [\![i]\!](w)(w)$ then the result follows directly by transitivity of $\geq$. So consider instead the case that $[\![\varphi]\!](w) < [\![i]\!](w)(w)$. But then

$$[\![\Box_i\varphi]\!](w) = \inf\{[\![\varphi]\!](w') \mid w' \in \mathcal{W} \text{ and } [\![\varphi]\!](w') < [\![i]\!](w)(w')\} \leq [\![\varphi]\!](w),$$

so $[\![\Box_i\varphi \to \varphi]\!](w) = 1 \geq [\![i]\!](w)(w)$.

For the other direction, suppose that F is not reflexive, so that there is some w such that $[\![i]\!](w)(w) < 1$. Pick a valuation such that for every w', $[\![P]\!](w') = [\![i]\!](w)(w')$. Then

$$\begin{aligned}[\![\Box_i P]\!](w) &= \inf\{[\![P]\!](w') \mid w' \in \mathcal{W} \text{ and } [\![P]\!](w') < [\![i]\!](w)(w')\} \\ &= \inf\emptyset \\ &= 1 \\ [\![\Box_i P \to P]\!](w) &= [\![P]\!](w) \\ &= [\![i]\!](w)(w) \\ &< 1.\end{aligned}$$

Thus, T is not valid for F. □

Proposition A.3. *It is* not *the case that whenever* $[\![i]\!]$ *is symmetric, the axiom B is valid for* **M**.

Proof. Consider the following simple model, in which $[\![P]\!](w) = \frac{1}{2}$.

We calculate

$$\begin{aligned}
[\![\Diamond_i P]\!](w) &= \sup\{[\![P]\!](w') \mid 1 - [\![i]\!](w)(w') < [\![P]\!](w')\} \\
&= \sup \emptyset \\
&= 0 \\
[\![\Box_i \Diamond_i P]\!](w) &= \inf\{[\![\Diamond_i P]\!](w') \mid [\![\Diamond_i P]\!](w') < [\![i]\!](w)(w')\} \\
&= \inf\{0\} \\
&= 0.
\end{aligned}$$

Consequently $[\![P]\!](w) > [\![\Box_i \Diamond_i P]\!](w)$ and so $[\![P \to \Box_i \Diamond_i P]\!](w) = [\![\Box_i \Diamond_i P]\!](w) = 0$. Since our model is clearly symmetric, this shows that symmetry does not imply the validity of B. □

Proposition A.4. *The axiom 4 is valid for a frame* F *iff* F *is transitive.*

Proof. We will first assume that F is transitive and show that 4 is valid for F. But prior to this, we confirm the following identity.

(A.1)

$$[\![\Box_i \Box_i \varphi]\!](w) = \inf\{[\![\varphi]\!](w'') \mid w'' \in \mathcal{W} \text{ and } \exists w' \in \mathcal{W} \,.\, [\![\varphi]\!](w'') < \min\{[\![i]\!](w)(w'), [\![i]\!](w')(w'')\}\}$$

We prove $\leq$ first. Let w' and w'' be given so that

$$[\![\varphi]\!](w'') < \min\{[\![i]\!](w)(w'), [\![i]\!](w')(w'')\}$$

and we will show that $[\![\Box_i \Box_i \varphi]\!] \leq [\![\varphi]\!](w'')$. This will establish that $[\![\Box_i \Box_i \varphi]\!]$ is less than or equal to the infimum on the right hand side.

In this case,

$$[\![\Box_i \varphi]\!](w') = \inf\{[\![\varphi]\!](v) \mid v \in \mathcal{W} \text{ and } [\![\varphi]\!](v) < [\![i]\!](w')(v)\} \leq [\![\varphi]\!](w'') < [\![i]\!](w)(w')$$

and hence

$$\begin{aligned}
[\![\Box_i \Box_i \varphi]\!](w) &= \inf\{[\![\Box_i \varphi]\!](v) \mid v \in \mathcal{W} \text{ and } [\![\Box_i \varphi]\!](v) < [\![i]\!](w)(v)\} \\
&\leq [\![\Box_i \varphi]\!](w') \\
&\leq [\![\varphi]\!](w'').
\end{aligned} \tag{A.2}$$

For the other direction, we use the definition of $[\![\Box_i \Box_i \varphi]\!]$ in equation (A.2) above. Let w' be given such that $[\![\Box_i \varphi]\!](w') < [\![i]\!](w)(w')$ and we will show that the right hand side of equation (A.1) is less than or equal to $[\![\Box_i \varphi]\!](w')$.

By definition,

$$[\![\Box_i \varphi]\!](w') = \inf\{[\![\varphi]\!](v) \mid v \in \mathcal{W} \text{ and } [\![\varphi]\!](v) < [\![i]\!](w')(v)\}$$

Since that infimum is strictly less than $[\![i]\!](w)(w')$, it follows that there is some u such that $[\![\varphi]\!](u) < [\![i]\!](w)(w')$ and $[\![\varphi]\!](u) < [\![i]\!](w')(u)$. Consequently, we see that

(A.3)

$$\inf\{[\![\varphi]\!](w'') \mid w'' \in \mathcal{W} \text{ and } \exists u \in \mathcal{W} \,.\, [\![\varphi]\!](w'') < \min\{[\![i]\!](w)(u), [\![i]\!](u)(w'')\}\} \leq [\![\varphi]\!](u) < [\![i]\!](w)(w').$$

Let v be given such that $[\![\varphi]\!](v) < [\![i]\!](w')(v)$. We will show that the right hand side of equation (A.1) is less than or equal to $[\![\varphi]\!](v)$, and thus (by the arbitrariness of v) is also less than or equal to $[\![\Box_i \varphi]\!](w')$. By inequality (A.3), we may assume that $[\![\varphi]\!](v) < [\![i]\!](w)(w')$. But then

$$[\![\varphi]\!](v) < \min\{[\![i]\!](w)(w'), [\![i]\!](w')(v)\}$$

and so clearly

$$\inf\{[\![\varphi]\!](w'') \mid \exists u \in \mathcal{W} \,.\, [\![\varphi]\!](w'') < \min\{[\![i]\!](w)(u), [\![i]\!](u)(w'')\}\} \leq [\![\varphi]\!](v).$$

With this tedious confirmation of equation (A.1) at hand, we proceed with our proof that 4 is valid for transitive frames. Let F be transitive, so that for every triple w, w', w'', we have

$$\min\{[\![i]\!](w)(w'), [\![i]\!](w')(w'')\} \leq [\![i]\!](w)(w').$$

Hence, for every w and w'', if there is a $u \in \mathcal{W}$ such that $[\![\varphi]\!](w'') < \min\{[\![i]\!](w)(u), [\![i]\!](u)(w'')\}$ then $[\![\varphi]\!](w'') < [\![i]\!](w)(w'')$. In other words, the set

$$\{[\![\varphi]\!](w'') \mid w'' \in \mathcal{W} \text{ and } \exists u \in \mathcal{W} \,.\, [\![\varphi]\!](w'') < \min\{[\![i]\!](w)(u), [\![i]\!](u)(w'')\}\}$$

is contained in the set

$$\{[\![\varphi]\!](w'') \mid w'' \in \mathcal{W} \text{ and } [\![\varphi]\!](w'') < [\![i]\!](w)(w'')\}\,.$$

The following calculation completes the proof that 4 is valid.

$$\begin{aligned}
[\![\Box_i\varphi]\!](w) &= \inf\{[\![\varphi]\!](w'') \mid w'' \in \mathcal{W} \text{ and } [\![\varphi]\!](w'') < [\![i]\!](w)(w'')\} \\
&\leq \inf\{[\![\varphi]\!](w'') \mid w'' \in \mathcal{W} \text{ and } \exists u \in \mathcal{W} \,.\, [\![\varphi]\!](w'') < \min\{[\![i]\!](w)(u), [\![i]\!](u)(w'')\}\} \\
&= [\![\Box_i\Box_i\varphi(w)]\!].
\end{aligned}$$

To complete the equivalence, we will show that if a frame F is not transitive, then 4 is not valid for F. Let F be given such that there exist $w, w', w'' \in \mathcal{W}$ and $i \in I$ satisfying

$$\min\{[\![i]\!](w)(w'), [\![i]\!](w')(w'')\} > [\![i]\!](w)(w'').$$

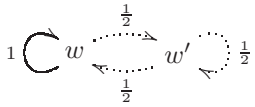
Pick a valuation such that $[\![P]\!](v) = [\![i]\!](w)(v)$ for all $v \in \mathcal{W}$. It is easy to check (as in the previous proofs) that $[\![\Box]\!][i]P(w) = 1$. We calculate

$$\begin{aligned}
[\![\Box_i\Box_i P]\!](w) &= \inf\{[\![P]\!](v) \mid v \in \mathcal{W} \text{ and } \exists u \in \mathcal{W} \,.\, [\![P]\!](v) < \min\{[\![i]\!](w)(u), [\![i]\!](u)(v)\}\} \\
&\leq [\![P]\!](w'') \\
&< 1
\end{aligned}$$

Thus, $[\![\Box_i P \to \Box_i\Box_i P]\!](w) = [\![\Box_i\Box_i P]\!] \leq [\![P]\!](w'') < 1$ and hence 4 is invalid. □

Proposition A.5. *It is* not *the case that whenever $[\![i]\!]$ is euclidean, axiom 5 holds.*

Proof. Consider the following example, in which $[\![P]\!](w) = \frac{1}{2}$ and $[\![P]\!](w') = 0$.



We calculate

$$\begin{aligned}
[\![\Diamond_i P]\!](w) &= \sup\{[\![P]\!](v) \mid v \in \mathcal{W} \text{ and } 1 - [\![i]\!](w)(v) < [\![P]\!](v)\} \\
&= \sup\left\{\frac{1}{2}\right\} \\
&= \frac{1}{2} \\
[\![\Diamond_i P]\!](w') &= \sup\{[\![P]\!](v) \mid v \in \mathcal{W} \text{ and } 1 - [\![i]\!](w')(v) < [\![P]\!](v)\} \\
&= \sup\emptyset \\
&= 0 \\
[\![\Box_i \Diamond_i P]\!](w') &= \inf\{[\![\Diamond_i P]\!](v) \mid v \in \mathcal{W} \text{ and } [\![\Diamond_i P]\!](v) < [\![i]\!](w)(v)\} \\
&= \inf\left\{0, \frac{1}{2}\right\} \\
&= 0.
\end{aligned}$$

Consequently, $[\![\Diamond_i P \to \Box_i \Diamond_i P]\!](w) = 0$. □

Appendix B. A comparison of the categories of PDL frames

In this section, we sketch the relationship between the various categories of PDL frames from Sections 2, 4 and 5.

We begin with the category of classical PDL frames from Section 2. It is well known that Kripke frames are coalgebras for the powerset functor, and it would be easy enough to fix a set Π_0 of actions and view classical PDL frames as coalgebras for the functor $X \mapsto (\mathcal{P}X)^{\Pi_0}$. This would be the category of PDL frames for a fixed set of actions, but instead we choose to consider the whole category of PDL frames[5] We fix the set Φ_0 throughout, however: varying the set of atomic propositions does not particularly interest us, but varying the set of actions allows one to vary the granularity with which one's options are described.

Thus, we are interested in the bifunctor

$$F_{\mathrm{cl}}(X, Y) = (\mathcal{P}X)^Y.$$

This functor is covariant in the first argument and contravariant in the second. This complicates matters somewhat: categories of coalgebras are traditionally defined for covariant functors. Nonetheless, extending the definition to bifunctors like F is reasonably straightforward.

Given a bifunctor $F : \mathbf{Set} \times \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$, let $\mathbf{Set}_F$ denote the category consisting of triples

$$\langle X,\, Y,\, \chi : X \to F(X, Y)\rangle\,.$$

A morphism $\langle X,\, Y,\, \chi : X \to F(X,Y)\rangle \to \langle Z,\, W,\, \zeta : Z \to F(Z,W)\rangle$ is a pair $\langle f,\, g\rangle$ where $f : X \to Z$ and $g : W \to Y$ (note the direction of g!) so that the diagram below commutes.

$$\begin{array}{ccc}
F(X,Y) & \xrightarrow{F(f,g)} & F(Z,W) \\
{\scriptstyle\chi}\uparrow & & \uparrow{\scriptstyle\zeta} \\
X & \xrightarrow[f]{} & Z
\end{array}$$

[5]In fact, the functor mapping a frame $\mathbf{F}$ to its set Π_0 of atomic actions is an op-fibration. The fibre over a set Π_0 is the category of PDL frames fixing the set of actions.

The objects of $\mathbf{Set}_{F_{cl}}$ are exactly the classical PDL frames. One must check, of course, that the morphisms are appropriate. Let

$$\mathbf{F} = \left\langle \mathcal{W},\, \Pi_0,\, [\![-]\!]^{\mathbf{F}} : \mathcal{W} \to (\mathcal{P}\mathcal{W})^{\Pi_0} \right\rangle,$$
$$\mathbf{F}' = \left\langle \mathcal{W}',\, \Pi_0',\, [\![-]\!]^{\mathbf{F}'} : \mathcal{W}' \to (\mathcal{P}\mathcal{W}')^{\Pi_0'} \right\rangle$$

be given together with $f : \mathcal{W} \to \mathcal{W}'$ and $g : \Pi' \to \Pi$. Unwinding the definition, one finds: $\langle f, g\rangle$ is a morphism iff for every $w \in \mathcal{W}$ and $m \in \Pi'$,

$$[\![m']\!]^{\mathbf{F}'}(f(w)) = \left\{ f(v) \;\middle|\; v \in [\![g(m')]\!]^{\mathbf{F}}(w) \right\}.$$

This condition is well-behaved with respect to the PDL semantics, but in order to show this, we need a bit of terminology.

A valuation $[\![-]\!]^{\mathbf{M}'} : \Phi_0 \to \mathcal{P}\mathcal{W}'$ induces a valuation $[\![-]\!]^{f} : \Phi_0 \to \mathcal{P}\mathcal{W}$, defined by $[\![P]\!]^{f} = f^{-1}([\![P]\!]^{M'})$. Thus, a model $\mathbf{M}' = \left\langle \mathcal{W}',\, \Pi_0',\, [\![-]\!]^{\mathbf{F}'},\, [\![-]\!]^{\mathbf{M}'} \right\rangle$ induces a model $\mathbf{M}^f$ defined by $\left\langle \mathcal{W},\, \Pi_0,\, [\![-]\!]^{\mathbf{F}},\, [\![-]\!]^{f} \right\rangle$, where $w \models^{\mathbf{M}^f} P$ iff $f(w) \models^{\mathbf{M}'} P$.

Let Π and Φ be the set of actions and formulas, resp., built from Π_0 and Φ_0, and let Π' and Φ' be the corresponding sets built from Π_0' and Φ_0. The map $g : \Pi_0' \to \Pi_0$ induces a function $\Pi' \to \Pi$ taking a composite action α' built from the atoms in Π_0' to the action $(\alpha')^g$ obtained by replacing each atom m' in α' with $g(m')$. This in turn induces a function $\Phi' \to \Phi$ by mapping a formula φ to the formula φ^g obtained by replacing actions α' occurring in φ with $g(\alpha')$.

With this introduced notation, we can now state the result regarding $\mathbf{Set}_{F_{\mathrm{cl}}}$. We sketch the proof, since the same basic reasoning is repeated for each of our categories.

Proposition B.1. *Let $\mathbf{F}$, $\mathbf{F}'$, f and g be given, as above. The following are equivalent.*

(1) $\langle f, g\rangle$ is a $\mathbf{Set}_{F_{\mathrm{cl}}}$ morphism.

(2) For every $\varphi \in \Phi'$, $w \in \mathcal{W}$ and valuation $[\![-]\!]^{\mathbf{M}'} : \Phi_0 \to \mathcal{P}\mathcal{W}'$,

$$[\![\varphi^g]\!]^{\mathbf{M}^f}(w) = [\![\varphi]\!]^{\mathbf{M}'}(f(w)).$$

In other words, $w \models^{\mathbf{M}^f} \varphi^g$ iff $f(w) \models^{\mathbf{M}} \varphi$.

Sketch. **(1)⇒(2):** By induction on the structure of propositions and actions, we jointly prove (2) and that for every $\alpha' \in \Pi'$ and $w \in \mathcal{W}$,

$$[\![\alpha']\!]^{\mathbf{F}'}(f(w)) = \left\{ f(v) \;\middle|\; v \in [\![(\alpha')^g]\!]^{\mathbf{F}}(w) \right\}.$$

(2)⇒(1): Define $[\![P]\!]^{\mathbf{M}'}(f(w)) = [\![m']\!]^{\mathbf{F}'}(f(w))$. Clearly, $f(w) \in [\![[m']P]\!]^{\mathbf{M}'}$ and hence $w \in [\![[g(m')]P]\!]^{\mathbf{M}^f}$. This implies

$$[\![m']\!]^{\mathbf{F}'}(f(w)) \supseteq \left\{ f(v) \;\middle|\; v \in [\![g(m')]\!]^{\mathbf{F}}(w) \right\}.$$

For the other direction, define $[\![Q]\!]^{\mathbf{M}'}(f(w)) = \{f(v) \mid v \in [\![g(m')]\!](w)\}$. It's easy to see that $w \in [\![[g(m')]Q]\!]^{\mathbf{M}^f}$ and hence $f(w) \in [\![[m']Q]\!]^{\mathbf{M}'}$. This rewrites to the desired inclusion $\subseteq$.

□

This completes our presentation of the category $\mathbf{Set}_{F_{\mathrm{cl}}}$ of classical PDL frames. The other two categories of PDL frames are constructed exactly the same way. We extend the subdistribution operator $\mathcal{D}$ to a functor by taking a function $f : X \to Y$ to the function $\mathcal{D}f : \mathcal{D}X \to \mathcal{D}Y$ taking a subdistribution p on X to the subdistribution on Y defined by

$$\mathcal{D}f(p)(y) = \sum_{f(x)=y} p(x).$$

The category of fuzzy PDL frames is just $\mathbf{Set}_{F_{\mathrm{fz}}}$, where $F_{\mathrm{fz}}(X,Y) = (\mathcal{D}X)^Y$ and that of minimal fuzzy PDL frames is $\mathbf{Set}_{F_{\mathrm{mfz}}}$, where $F_{\mathrm{mfz}}(X,Y) = (\mathcal{P}\mathcal{D}X)^Y$. In both cases, one can prove the analog to Proposition B.1.

For each set X, there is a function $\{-\}_X : X \to \mathcal{P}X$ taking $x \in X$ to the singleton $\{x\} \in \mathcal{P}X$. These components define a natural transformation $\{-\} : 1 \to \mathcal{P}$. The transformation induces a functor $I : \mathbf{Set}_{F_{\mathrm{fz}}} \to \mathbf{Set}_{F_{\mathrm{mfz}}}$ taking an F_{fz}-coalgebra $\langle X, Y, \alpha : X \to F_{\mathrm{fz}}(X,Y)\rangle$ to the F_{mfz}-coalgebra with structure map

$$X \xrightarrow{\alpha} (\mathcal{D}X)^Y \xrightarrow{(\{-\}_{\mathcal{D}X})^Y} (\mathcal{P}\mathcal{D}X)^Y$$

This structure map takes $x \in X$ and $y \in Y$ to the singleton $\{\alpha(x)(y)\}$. This functor is injective on objects, full and faithful—in other words, the category of fuzzy PDL frames is a full subcategory of the category of minimal fuzzy PDL frames. A minimal fuzzy PDL frame $\langle \mathcal{W}, \Pi_0, [\![-]\!]\rangle$ is in this subcategory iff $[\![-]\!]$ factors through $(\{-\}_{\mathcal{D}\mathcal{W}})^{\Pi_0}$.

Moreover, the inclusion preserves and reflects satisfaction, in the sense that for every fuzzy PDL model $\mathbf{M}$, world w in $\mathbf{M}$ and formula φ (omitting non-deterministic choice and iteration, as in Section 4), we have

$$[\![\varphi]\!]^{\mathbf{M}}(w) = [\![\varphi]\!]^{I(\mathbf{M})}(w).$$

There is a similar natural transformation $\{-\} : 1 \to \mathcal{D}$ that acts on components X by mapping $x \in X$ to the singleton $\{x\}_X$, i.e. to that subdistribution defined by

$$\{x\}_X(y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{else.} \end{cases}$$

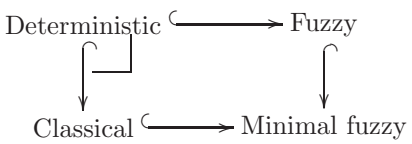
This induces a functor $\mathbf{Set}_{F_{\mathrm{cl}}} \to \mathbf{Set}_{F_{\mathrm{mfz}}}$, mapping a coalgebra $\langle X, Y, \alpha : X \to F_{\mathrm{cl}}(X,Y)\rangle$ to the F_{mfz}-coalgebra with structure map

$$X \xrightarrow{\alpha} (\mathcal{P}X)^Y \xrightarrow{(\mathcal{P}\{-\}_X)^Y} (\mathcal{P}\mathcal{D}X)^Y$$

This establishes the category of classical PDL frames as a full subcategory of the category of minimal fuzzy PDL frames. Again, the inclusion preserves and reflects satisfaction, but in this case, we must restrict ourselves to crisp valuations (since the semantics for classical PDL models requires this).

Thus, we have two full subcategories of the category of minimal fuzzy PDL frames. It is natural to ask what the pullback (intersection) of these two subcategories is. Again, we have a very natural answer: it is the category of deterministic PDL frames, i.e. the category $\mathbf{Set}_{F_{\mathrm{det}}}$, where $F_{\mathrm{det}}(X,Y) = (1+X)^Y$. This category was mentioned in Section 4.

We summarize the situation in the following pullback diagram, in which each arrow is a full and faithful inclusion.



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