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# Modal Operators and the Formal Dual of Birkhoff's Completeness Theorem

STEVE AWODEY[1] and JESSE HUGHES[2†]

[1] *Dept. Philos., Carnegie Mellon University, Pittsburgh, PA 15213*
[2] *Dept. Comp. Sci., Univ. Nijmegen, 6525 ED Nijmegen, The Netherlands*

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We present the dual to Birkhoff's variety theorem in terms of predicates over the carrier of a cofree coalgebra (i.e., in terms of "coequations"). We then discuss the dual to Birkhoff's completeness theorem, showing how closure under deductive rules dualizes to yield two modal operators acting on coequations. We discuss the properties of these operators and show that they commute, and we prove as main result the invariance theorem, which is the formal dual of Birkhoff's completeness theorem.

## 1. Introduction

The topic of dualizing Birkhoff's variety theorem (Birkhoff, 1935) was first raised in (Jacobs, 1995), and an early dual appears in (Rutten, 2000) (for coalgebras over **Set**). Since then, a number of authors have extended these results, providing the notions of coequations and implications between them, weakening the assumptions on the base category and endofunctor and dualizing Birkhoff's deductive completeness theorem (Birkhoff, 1935). We will briefly survey these results here.

In what follows, we consider a category $\mathcal{E}$ and an endofunctor $\Gamma : \mathcal{E} \twoheadrightarrow \mathcal{E}$. We denote the category of $\Gamma$-coalgebras $\mathcal{E}_\Gamma$ and the category of $\Gamma$-algebras $\mathcal{E}^\Gamma$. We abuse notation and use $U$ to denote both forgetful functors $\mathcal{E}_\Gamma \twoheadrightarrow \mathcal{E}$ and $\mathcal{E}^\Gamma \twoheadrightarrow \mathcal{E}$. We omit $U$ when convenient, writing $A$ for $U\langle A, \alpha\rangle$ and just $p$ for $Up$. We use $H : \mathcal{E} \twoheadrightarrow \mathcal{E}_\Gamma$ to denote the right adjoint to $U : \mathcal{E}_\Gamma \twoheadrightarrow \mathcal{E}$, if it exists, and $F : \mathcal{E} \twoheadrightarrow \mathcal{E}^\Gamma$ to denote the left adjoint to $U : \mathcal{E}^\Gamma \twoheadrightarrow \mathcal{E}$, if it exists. Given $C \in \mathcal{E}$, we call $HC$ the *cofree coalgebra over* $C$, just as $FX$ is called the *free algebra over* $X$.

In (Gumm and Schröder, 2001; Gumm, 2001b; Gumm, 1999; Gumm, 2001a), the authors define a coequation as an element $c \in UHC$ of a cofree coalgebra, just as an equation is a pair of elements $\langle t_1, t_2\rangle \in UFX \times UFX$ of a free algebra. Coequation satisfaction can then be stated in terms of omission: A coalgebra $\langle A, \alpha\rangle$ satisfies $c$ just in case for every $\mathbb{G}$-homomorphism $p : \langle A, \alpha\rangle \twoheadrightarrow HC$, $c$ is not in the image of $p$. In contrast, in (Kurz, 1999; Kurz, 2000) and also (Awodey and Hughes, 2000; Hughes, 2001), the authors take

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the point of view that a coequation is not an element of $UHC$, but a predicate (i.e., subobject) over $UHC$. Here, coequation satisfaction is given as a projectivity condition: $\langle A, \alpha\rangle$ satisfies a coequation $\varphi \subseteq UHC$ just in case every $\mathbb{G}$-morphism $\langle A, \alpha\rangle \longrightarrow HC$ factors through $\varphi$. The two definitions are essentially equivalent in the standard setting $\mathcal{E} = \mathbf{Set}$, since a coalgebra satisfies a coequation $c \in UHC$ in the former sense just in case it satisfies $UHC \setminus \{c\} \subseteq UHC$ in the latter sense.

It is also in (Gumm and Schröder, 2001) that we first find the invariance theorem over **Set**, which arises as the formal dual of Birkhoff's completeness theorem. It is this theorem which forms the focus of the current paper. We view Birkhoff's theorem as establishing an equality between two closure operators on sets of equations. More precisely, fix a signature $\Sigma$ and a set $X$ of variables and consider a set of $\Sigma$-equations $S$ over $X$. If $\mathsf{Ded}(S)$ denotes the deductive closure of $S$ under Birkhoff's equational calculus and $\mathsf{Th}\,\mathsf{Mod}(S)$ the equational theory of the models of $S$, then the completeness theorem asserts

$$\mathsf{Ded}(S) = \mathsf{Th}\,\mathsf{Mod}(S).$$

In ibid, we find the basic pieces of the invariance theorem. Namely, we see a proof that the endomorphism invariant subcoalgebras of cofree coalgebras $UHC$ are in bijective correspondence with the covarieties definable by coequations over $C$. Furthermore, we see that the endomorphism invariant subcoalgebras are *generating* coequations in the sense that they are the formal dual of maximal sets of equations satisfied by a collection of algebras, i.e., *equational theories.*

We revisit these results and make explicit two interior operators which were implicit in the presentation in (Gumm and Schröder, 2001), so that the presentation of the invariance theorem mimics the familiar development of the completeness theorem as an equation between closure operators. Specifically, we show that the dual of closure of sets of equations under reflexivity, symmetry, transitivity and term formation is the $\Box$ operator first investigated in (Jacobs, 1999). Closure under substitution of terms for variables dualizes to yield the $\boxslash$ operator, introduced here. The $\boxslash$ operator is an **S4**-operator, while $\Box$ is **S4** if the forgetful functor $U : \mathcal{E}_{\mathbb{G}} \longrightarrow \mathcal{E}$ preserves binary meets. Furthermore, $\Box$ preserves pullbacks along homomorphisms, while $\boxslash$ does not.

This presentation makes explicit use of the coequations-as-predicates viewpoint. We consider a coequation $\varphi$ over $C$ as a predicate over $UHC$ (via the regular subobject fibration). Thus, coequations come equipped with the usual logical constructions via the connectives $\wedge$, $\rightarrow$, etc. (depending, of course, on the structure of $\mathcal{E}$), just in virtue of the fact that coequations are predicates. We can view the work here as augmenting these usual constructions by adding two **S4** operators, $\Box$ and $\boxslash$, for coequations and showing that the coequations "open" with respect to these operators are in bijective correspondence to covarieties definable by coequations over $C$.

In (Gumm and Schröder, 2001) and (Roşu, 2000), we find the first discussion of "complete" or "behavioral" covarieties (called "sinks" in (Roşu, 2000)). These covarieties are definable by coequations over one "color" or, equivalently, are the covarieties closed under total bisimulations. The work on coalgebraic specifications in (Rothe et al., 2001), for instance, involves giving models for classes in an object oriented language as behavioral covarieties in an appropriate category of coalgebras. Hence, we can understand this approach in terms of coequations over a single color. These coequations are dual to variable-free equations for a class of universal algebras, and so one has the idea that there is much more expressive power to exploit in the theory of coequations. Indeed, from (Kurz, 2001), we see that such coequations correspond to modal formulas with variables and thus allow the richer expressiveness such formulas have over variable-free formulas (see, for instance, (Blackburn et al., 2001)). We provide examples of coequations here which illustrate some of the expressive power available when one moves from behavioral covarieties to covarieties in general.

We also note that when $\varphi$ is a coequation over 1, i.e., a coequation defining a behavioral covariety, then $\boxslash\varphi = \varphi$. So, for behavioral covarieties, one of the two **S4** operators is trivial. Note that in the dual situation, this amounts to the observation that if $S$ is a collection of variable-free equations, then closure of $S$ under substitution of terms for variables yields $S$ again!

We begin with some technical preliminaries in Section 2. In Section 3, we summarize the dual of Birkhoff's variety theorem, introducing the relevant terminology and results. In Section 4, we generalize the covariety theorem to accommodate quasi-covarieties and conditional coequations. Section 5 is a categorical presentation of Birkhoff's deductive completeness theorem and its dual, the invariance theorem. We discuss the well-known greatest subcoalgebra operator, $\Box$, in Section 6 and show that it is an **S4** modal operator that commutes with pullbacks along homomorphisms. In Section 7, we introduce a second **S4** operator, $\boxslash$, taking a coequation to its largest invariant sub-coequation. This allows an easy proof of the invariance theorem in terms of the operators $\Box$ and $\boxslash$ in Section 8.

This work forms part of the second author's doctoral dissertation, written under the supervision of Dana S. Scott and the first author. Scott suggested research into the dual of Birkhoff's theorems, and that research and the presentation found here benefited from many conversations with him. We also benefited from conversations with Jiří Adámek, who pointed us to the Banaschewski and Herrlich article, Peter Gumm, Bart Jacobs, Alexander Kurz, Tobias Schröder and the advice of anonymous reviewers, who have pointed us to the work of Andréka and Németi.

## 2. Preliminaries

An arrow $i\colon A \rightarrowtail B$ is a *regular mono* just in case it is the equalizer for some pair of maps $B \rightrightarrows \bullet$. (Dually, a regular epi is a coequalizer for some pair of arrows.) The partial order of isomorphism classes of regular monos with codomain $B$ is denoted $\mathsf{Sub}(B)$, and we call the elements of $\mathsf{Sub}(B)$ the *(regular) subobjects* of $B$. We denote the subobject $1_B\colon B \rightarrowtail B$ by $\top$. If $\mathcal{E}$ has pullbacks along regular monos, then $\mathsf{Sub}(B)$ clearly has binary meets. We say that $\mathcal{E}$ is *regularly well-powered* if, for every $B \in \mathcal{E}$, $\mathsf{Sub}(B)$ is a set.

Throughout, we denote regular monos by $\rightarrowtail$ and epis by $\twoheadrightarrow$. We say that $\mathcal{E}$ has epi-regular mono factorizations just in case every map $f\colon A \to B$ in $\mathcal{E}$ factors into an epi followed by a regular mono, as in the diagram below.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ C \arrow[ur, tail] & \end{tikzcd}$$

Such factorizations are unique up to isomorphism. We denote the subobject $C$ of $B$ (properly, the equivalence class of $C \rightarrowtail B$) by $\mathsf{Im}(f)$. Some authors use $f(C)$ for $\mathsf{Im}(f)$.

If $\mathcal{E}$ has epi-regular mono factorizations and coproducts, then each $\mathsf{Sub}(B)$ has arbitrary joins. Given $\{j_i : C_i \rightarrowtail B\}_{i \in I} \in \mathsf{Sub}(B)$, then $\bigvee C_i$ is the image of the map

$$\coprod C_i \xrightarrow{[j_i]} B.$$

**Remark 2.1.** Throughout, our assumption that $\mathcal{E}$ has epi-regular mono factorizations could be replaced by assuming any factorization system $\langle \mathcal{H}, \mathcal{S} \rangle$ where $\mathcal{E}$ is $\mathcal{S}$-well-powered in the evident sense. See (Borceux, 1994) for a presentation of factorization systems and also (Adámek and Koubek, 1977) for a discussion of factorization systems in the dual category, $\mathcal{E}^{\mathbb{G}}$. We stick with epi-regular mono factorizations here since there is a natural relationship between regular epis and sets of equations in the algebraic setting and we are interested in dualizing the algebraic Birkhoff theorems.

Suppose $\mathcal{E}$ has epi-regular mono factorizations and let $f : A \to B$ be given. The arrow $f$ induces a functor $\mathsf{Sub}(A) \to \mathsf{Sub}(B)$, which we denote $\exists_f$ (some authors abuse notation and denote the induced morphism $f$). Namely, given a subobject $i : C \rightarrowtail A$ of $A$, we define $\exists_f(C) = \mathsf{Im}(f \circ i)$, as in the diagram below.

$$\begin{tikzcd} C \arrow[r, two heads] \arrow[d, tail, "i"'] & \exists_f(C) \arrow[d, tail] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Note that $\exists_f(\top) = \mathsf{Im}(f)$.

If $\mathcal{E}$ also has pullbacks, then each $f : A \to B$ also induces a functor

$$f^* : \mathsf{Sub}(B) \longrightarrow \mathsf{Sub}(A)$$

(sometimes denoted $f^{-1}$) given by taking a subobject $j : D \rightarrowtail B$ to the pullback shown below.

$$\begin{tikzcd} f^*D \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, tail] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

In this case, we have an adjunction $\exists_f \dashv f^*$.

A coalgebra for an endofunctor $\Gamma : \mathcal{E} \to \mathcal{E}$ is a pair $\langle A, \alpha \rangle$, where $A \in \mathcal{E}$ and $\alpha : A \to \Gamma A$. A $\Gamma$-homomorphism $f : \langle A, \alpha \rangle \to \langle B, \beta \rangle$ is a map $fAB$ such that $\Gamma f \circ \alpha = \beta \circ f$. The category of $\Gamma$-coalgebras is denoted $\mathcal{E}_\Gamma$. The evident forgetful functor $\mathcal{E}_\Gamma \to \mathcal{E}$ is denoted $U$.

A *comonad* on $\mathcal{E}$ is a triple $\mathbb{G} = \langle G,\, \varepsilon,\, \delta \rangle$, where $G : \mathcal{E} \to \mathcal{E}$ and $\varepsilon : G \Rightarrow 1$ and $\delta : G \Rightarrow GG$ are natural transformations such that $\delta_G \circ \delta = G\delta \circ \delta$ and $\varepsilon_G \circ \delta = 1_G = G\varepsilon \circ \delta$. The category $\mathcal{E}_\mathbb{G}$ of coalgebras for the *comonad* $\mathbb{G}$ consists of coalgebras $\langle A,\, \alpha : A \to GA \rangle$ for the functor $G$ such that $\delta_A \circ \alpha = G\alpha \circ \alpha$ and $\varepsilon_A \circ \alpha = 1_A$. It is a full subcategory of $\mathcal{E}_G$, the category of coalgebras for the functor $G$. The forgetful functor $U : \mathcal{E}_\mathbb{G} \to \mathcal{E}$ (note the overloaded use for $U$!) has a right adjoint $H : \mathcal{E} \to \mathcal{E}_\mathbb{G}$ (again abusing notation) given by

$$H(C) = \langle GC,\, \delta_C \rangle.$$

We say that a functor $K : \mathcal{D} \to \mathcal{E}$ is *comonadic* just in case there is a comonad $\mathbb{G}$ on $\mathcal{E}$ and an equivalence of categories $\mathcal{D} \to \mathcal{E}_\mathbb{G}$ such that the diagram below commutes.

$$\begin{tikzcd}
\mathcal{D} \arrow[r] \arrow[d, "K"'] & \mathcal{E}_\mathbb{G} \arrow[dl, "U"] \\
\mathcal{E} &
\end{tikzcd}$$

Given any endofunctor $\Gamma : \mathcal{E} \to \mathcal{E}$, the forgetful functor $U : \mathcal{E}_\Gamma \to \mathcal{E}$ is comonadic iff $U$ has a right adjoint (see (Jacobs, 1995)). Such functors $\Gamma$ are called *covarietors* in (Adámek and Porst, 2001). (Note: it is *not* the case for an arbitrary functor $K : \mathcal{D} \to \mathcal{E}$ that $K$ is comonadic iff $K$ has a right adjoint.)

Let $S$ be a collection[†] of arrows. We say that an object $A$ is *$S$-projective* just in case, for every $f : B \to C$ in $S$, and every $g : A \to C$, there is a (not necessarily unique[‡]) $h : A \to B$ such that $g = f \circ h$. We denote the full subcategory of $S$-projective objects by $S$ - **Proj**.

The dual of $S$-projectivity is *$S$-injectivity.* In particular, we say that an object $A$ is *regular mono-injective* (hereafter, just *regular injective*) just in case, for every regular subobject $B \rightarrowtail C$ and every $f : B \to A$, there is a (not necessarily unique) map $g : C \to A$ such that the diagram below commutes.

$$\begin{tikzcd}
C \arrow[r, dotted, "g"] & A \\
B \arrow[u, tail] \arrow[ur, "f"'] &
\end{tikzcd}$$

We say that a category $\mathcal{E}$ *has enough regular injectives* if, for every object $E \in \mathcal{E}$, there is a regular injective $A$ such that $E$ is a regular subobject of $A$.

[†] When we use the word *collection*, we allow that it is a proper class. We often abuse set notation and adopt it for classes in what follows.

[‡] In fact, for the collections $S$, namely collections of (regular) monos, in which we are interested, uniqueness is trivial. Our previous presentations of this material used this fact, and developed the theory that follows in terms of orthogonality, as in (Borceux, 1994, Volume 2). Here, we prefer to ensure that our presentation more closely follows that of (Banaschewski and Herrlich, 1976), (Németi and Sain, 1981), etc.

## 3. The dual of Birkhoff's variety theorem

We begin with a brief summary of the dual of Birkhoff's variety theorem. This section summarizes the work found in (Awodey and Hughes, 2000), which can be viewed as a generalization of (Rutten, 2000) and (Gumm and Schröder, 2001). A similar account of the covariety theorem can be found in (Kurz, 2000), and a similar categorical approach to the variety theorem for categories of algebras can be found in (Banaschewski and Herrlich, 1976) and various articles of Andréka and Németi (see bibliography). The reader familiar with previous treatments of the covariety and quasi-covariety theorems may skip ahead to Section 5.

The following definitions come from (Awodey and Hughes, 2000; Hughes, 2001).

**Definition 3.1.** We say that a category $\mathcal{E}$ is *quasi-co-Birkhoff* if it is regularly well-powered, cocomplete and has epi-regular mono factorizations. If, in addition, $\mathcal{E}$ has enough regular injectives, then $\mathcal{E}$ is *co-Birkhoff*.

A full subcategory of a quasi-co-Birkhoff category is a *quasi-covariety* iff it is closed under coproducts and codomains of epis. A quasi-covariety of a co-Birkhoff category is a *covariety* iff it is also closed under regular subobjects.

The following theorem can be found in (Awodey and Hughes, 2000) or (Hughes, 2001). See also (Kurz, 1999; Kurz, 2000) and see (Banaschewski and Herrlich, 1976) for a presentation of the dual theorem.

**Theorem 3.2.** If $\mathcal{E}$ is a co-Birkhoff category, then $\mathbf{V}$ is a covariety iff $\mathbf{V} = S\text{-}\mathbf{Proj}$ for some collection $S$ of regular monos with regular injective codomains.

One can show that, if $\mathbb{G} = \langle G, \varepsilon, \delta \rangle$ is a comonad on a quasi-co-Birkhoff category $\mathcal{E}$ and $G$ preserves regular monos, then $\mathcal{E}_{\mathbb{G}}$ inherits the epi-regular mono factorizations from $\mathcal{E}$. We use this fact to prove the following.

**Theorem 3.3.** Let $\mathbb{G} = \langle G, \varepsilon, \delta \rangle$ be a comonad on a (quasi-)co-Birkhoff category $\mathcal{E}$ and suppose that $G$ preserves regular monos. Then $\mathcal{E}_{\mathbb{G}}$ is (quasi-)co-Birkhoff.

In fact, Theorem 3.3 applies more generally than stated. If $\mathcal{E}$ is a quasi-co-Birkhoff category and $\Gamma$ is any endofunctor that preserves regular monos, then the category $\mathcal{E}_\Gamma$ of coalgebras for the endofunctor $\Gamma$ is quasi-co-Birkhoff. In other words, Theorem 3.3 applies to categories of coalgebras for an endofunctor as well as to categories of coalgebras for a monad.

**Example 3.4.** Let $\mathbb{G} = \langle G, \varepsilon, \delta \rangle$ be a comonad on $\mathcal{E}$ and suppose that $\mathcal{E}$ is co-Birkhoff and $G$ preserves regular monos, so that the category $\mathcal{E}_G$ of coalgebras for the endofunctor $G$ is co-Birkhoff. The category $\mathcal{E}_{\mathbb{G}}$ of coalgebras for the comonad $\mathbb{G}$ forms a covariety in the category $\mathcal{E}_G$ of coalgebras for the functor $G$. Indeed, since both forgetful functors $\mathcal{E}_{\mathbb{G}} \twoheadrightarrow \mathcal{E}$ and $\mathcal{E}_G \twoheadrightarrow \mathcal{E}$ create coproducts, $\mathcal{E}_{\mathbb{G}}$ is closed under coproducts. One uses the fact that (under our assumptions) these functors also preserve and reflect epis and regular monos to show that $\mathcal{E}_{\mathbb{G}}$ is also closed under codomains of epis and regular subobjects. We omit the details.

Throughout what follows, we state our theorems in terms of coalgebras for a comonad, although we often indicate when the theorem applies to coalgebras for an endofunctor as well. Recall that, whenever an endofunctor $\Gamma : \mathcal{E} \longrightarrow \mathcal{E}$ is a covarietor (i.e., $\mathcal{E}_\Gamma$ has cofree coalgebras), then $\mathcal{E}_\Gamma$ is isomorphic to a category of coalgebras for a comonad. Although in ibid, the authors show that the assumption that $\Gamma$ is a covarietor is not crucial for the covariety theorem, we stick to the more familiar and convenient territory of coalgebras for a comonad here.

Theorem 3.3 ensures that we may apply Theorem 3.2 to categories $\mathcal{E}_{\mathbb{G}}$ of coalgebras for a comonad $\mathbb{G} = \langle G, \varepsilon, \delta \rangle$, provided that $G$ preservers regular monos. In order to interpret the result, we introduce the notion of coequation.

**Definition 3.5.** Let $C \in \mathcal{E}$ be regular injective, so that the cofree coalgebra $HC = \langle GC, \delta_C \rangle$ is also regular injective. A *coequation over* $C$ is a regular subobject $\varphi \leq GC (= UHC)$. We say that a coalgebra $\langle A, \alpha \rangle$ *satisfies* $\varphi$ (written $\langle A, \alpha \rangle \models \varphi$) if, for every homomorphism

$$p : \langle A, \alpha \rangle \longrightarrow HC$$

(equivalently, every "coloring" $A \longrightarrow C$), there is a unique map

$$\widetilde{p} : A \longrightarrow \varphi$$

making the diagram below commute.

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[dr, dotted, "\widetilde{p}"'] & GC \\ & \varphi \arrow[u, tail] \end{tikzcd}$$

If $\mathbf{V}$ is a class of coalgebras, we write $\mathbf{V} \models \varphi$ just in case each $\langle A, \alpha \rangle \in \mathbf{V}$ satisfies $\varphi$.

In other words, $\langle A, \alpha \rangle \models \varphi$ if, for every homomorphism $p : \langle A, \alpha \rangle \longrightarrow HC$, we have $\mathsf{Im}(p) \leq \varphi$, or, equivalently, $\top \leq p^* \varphi$ (assuming $\mathcal{E}$ has pullbacks).

We similarly define, for each $p : \langle A, \alpha \rangle \longrightarrow HC$,

$$\langle A, \alpha \rangle \models \varphi(p) \text{ iff } \mathsf{Im}(p) \leq \varphi,$$

so $\langle A, \alpha \rangle \models \varphi$ just in case $\langle A, \alpha \rangle \models \varphi(p)$ for every homomorphism $p : \langle A, \alpha \rangle \longrightarrow HC$.

A coequation $\varphi$ over $C$ can be viewed as a predicate over $GC$. Thus, if $\mathsf{Sub}(GC)$ is a Heyting algebra, we can construct coequations $\varphi \wedge \psi$, $\varphi \to \psi$, etc., and so we see that coequations over $C$ come with a natural structure. Continuing this interpretation, if $\varphi, \psi \in \mathsf{Sub}(GC)$, we often write $\varphi \vdash \psi$ to mean $\varphi \leq \psi$. It is easy to see that, if $\varphi \vdash \psi$ and $\langle A, \alpha \rangle \models \varphi$, then $\langle A, \alpha \rangle \models \psi$.

**Remark 3.6.** This definition of coequation comes from a straightforward dualization of the abstraction of equational definability first explored in (Banaschewski and Herrlich, 1976). In this perspective, a set of equations generalizes to a (regular) epi with regular projective domain. Equational satisfaction corresponds to injectivity with respect to such epis. Dualizing this abstract approach leads directly to Theorem 3.2. Specializing Banaschewski and Herrlich's work to categories of algebras, sets of equations over $X$

correspond to regular epis $UFX \twoheadrightarrow Q$ with domain the carrier of the free algebra $FX$. Dualizing yields regular monos into cofree coalgebras.

Suppose $\mathcal{E}$ has pullbacks. If we view coequations $\varphi$ over $C$ as predicates of a variable $x$ of type $GC$, one may interpret pullback of coequations along homomorphisms

$$p\colon\langle A,\, \alpha\rangle \longrightarrow GC$$

as substitution of $p(y)$ (where $y$ is a variable of type $A$) for $x$, i.e., $p^*\varphi = \varphi[p(y)/x]$. Thus, $\langle A,\, \alpha\rangle \models \varphi$ just in case, for every homomorphism $p$, we have $\top \vdash \varphi[p(y)/x]$.

**Remark 3.7.** In the case of equations, one can easily distinguish between single equations and sets of equations. Gumm makes a similar distinction between single coequations and sets of coequations in (Gumm, 2001a), by interpreting coequation satisfaction as an exclusionary condition. Explicitly, in Gumm's terms, a coequation is an element $c$ of a cofree coalgebra $UHC$, and we say $\langle A,\, \alpha\rangle \models c$ just in case $\langle A,\, \alpha\rangle \models UHC \setminus \{c\}$ (in our terms). We prefer to keep the definition of satisfaction above, in keeping with our view of coequations as predicates. Hence, we do not distinguish between single coequations and sets of coequations.

This notion of coequation allows a more familiar statement of the dual of Birkhoff's variety theorem.

**Theorem 3.8.** Suppose $\mathcal{E}$ is co-Birkhoff and $\mathbb{G}$ preserves regular monos. Then a full subcategory $\mathbf{V}$ of $\mathcal{E}_{\mathbb{G}}$ is a covariety iff there is a collection $S$ of coequations such that for all $\langle A,\, \alpha\rangle$,

$$\langle A,\, \alpha\rangle \in \mathbf{V} \text{ iff } \forall \varphi \in S\ \langle A,\, \alpha\rangle \models \varphi.$$

See (Rutten, 2000) or (Gumm and Schröder, 2001), for instance, where Theorem 3.8 is proved for coalgebras over **Set**. A proof of this theorem in a more general setting can be found in (Hughes, 2001) or (Kurz, 2000).

Note that there's still some work to be done to prove this theorem. It is not an immediate corollary to Theorems 3.2 and 3.3. The immediate corollary is that there is a collection $T$ of subcoalgebras of regular injective coalgebras such that $\mathbf{V} = T\text{-}\mathbf{Proj}$. In Theorem 3.8, on the other hand, we have a collection $S$ of subobjects of cofree coalgebras over regular injective $C$, that is,

$$S = \{\varphi_i \leq UHC_i\}_{i \in I}$$

for some indexed set $I$. We omit the proof of this theorem here, as it has been adequately covered elsewhere. See, for instance, (Kurz, 2000) or (Hughes, 2001) for details. The key step in showing that cofree coalgebras over regular injective $C$ suffice is applying the assumption that $\mathcal{E}$ has enough regular injectives. This implies that every coalgebra $\langle A,\, \alpha\rangle$ is a subcoalgebra of a cofree coalgebra over regular injective $C$. One finishes by showing that there is a coequation $\varphi$ over $C$ such that $\langle A,\, \alpha\rangle \in \mathbf{V}$ iff $\langle A,\, \alpha\rangle \models \varphi$.

**Remark 3.9.** A **Set**-functor $\Gamma\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ is called *bounded* by a cardinal $\kappa$ if, for every $\Gamma$ coalgebra $\langle A,\, \alpha\rangle$ and every $a \in A$, there is a subcoalgebra $\langle D,\, \delta\rangle \leq \langle A,\, \alpha\rangle$ such that

$a \in D$ and $\mathbf{card}(D) \leq \kappa$ (where $\mathbf{card}(D)$ is the cardinality of $D$). We say that a comonad $\mathbb{G} = \langle G, \varepsilon, \delta \rangle$ is bounded by $\kappa$ if the functor $G$ is bounded by $\kappa$. One can show that, if $\mathbb{G}$ is bounded by $\kappa$, then for each covariety $\mathbf{V}$ of $\mathcal{E}_{\mathbb{G}}$, there is a single coequation $\varphi$ over $\kappa$ (i.e., a subset $\varphi \subseteq UH\kappa$) such that

$$\langle A, \alpha \rangle \in \mathbf{V} \text{ iff } \langle A, \alpha \rangle \models \varphi.$$

This further refinement corresponds to showing that a variety of algebras for a signature is definable by a set of equations over a *countable* set $X$ of variables. Here, the assumption that $\mathbb{G}$ is bounded takes the place of the assumption that each function symbol has finite arity.

See (Rutten, 2000) or (Gumm and Schröder, 2001) for a proof of this refinement of Theorem 3.8. See (Hughes, 2001) for a generalization of bounded functors to categories other than **Set** and the corresponding theorem. See also (Adámek and Porst, 2001) for a discussion of bounded functors, including a proof that for functors $\Gamma : \mathbf{Set} \longrightarrow \mathbf{Set}$, $\Gamma$ is bounded iff $\Gamma$ is accessible.

**Example 3.10.** Fix a set of "inputs", $\mathcal{I}$ and let $\Gamma : \mathbf{Set} \longrightarrow \mathbf{Set}$ be defined by

$$\Gamma S = (\mathcal{P}_{\mathsf{fin}} S)^{\mathcal{I}},$$

where $\mathcal{P}_{\mathsf{fin}}$ is the covariant finite powerset functor. A $\Gamma$-coalgebra $\langle S, \sigma \rangle$ can be regarded as a non-deterministic automaton over $\mathcal{I}$, where the structure map gives the transition function. Explicitly, for each state $s \in S$ and each input $i \in \mathcal{I}$, we write

$$s \xrightarrow{i} s'$$

just in case $s' \in \sigma(s)(i)$.

The deterministic automata are those automata $\langle S, \sigma \rangle$ such that, for each $s \in S$ and each $i \in \mathcal{I}$, there is at most one $s'$ such that $s \xrightarrow{i} s'$. Let $\mathcal{D}et$ denote the class of deterministic automata, so $\mathcal{D}et \subseteq \mathbf{Set}_\Gamma$. It is easy to see that $\mathcal{D}et$ is a covariety in $\mathbf{Set}_\Gamma$.

In fact, one can show that there is a coequation $\varphi$ over 2 colors that defines $\mathcal{D}et$. Namely, let $H : \mathbf{Set} \longrightarrow \mathbf{Set}_\Gamma$ be the right adjoint to $U : \mathbf{Set}_\Gamma \longrightarrow \mathbf{Set}$ and define $\varphi \subseteq UH2$ by

$$\varphi = \{x \in UH2 \mid \forall i \in \mathcal{I}\, \forall y, z \in \delta_2(x)(i) \,.\, \varepsilon_2(y) = \varepsilon_2(z)\},$$

where $\delta_2 : UH2 \longrightarrow \Gamma UH2$ is the structure map for $H2$ and $\varepsilon : UH \Longrightarrow 1$ is the counit of the adjunction $U \dashv H$. Then, it is easy to show that

$$\langle A, \alpha \rangle \models \varphi \text{ iff } \langle A, \alpha \rangle \in \mathcal{D}et\,.$$

Indeed, one can show that a coequation over 1 color cannot define $\mathcal{D}et$. As shown in (Gumm and Schröder, 2001), a coequation $\psi$ over 1 is "behavioral", in the sense that, if $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ are related by a total bisimulation, then $\langle A, \alpha \rangle \models \psi$ just in case $\langle B, \beta \rangle \models \psi$. That is, the covariety $\mathbf{V}$ defined by $\psi$ is closed under total bisimulations (i.e., if $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ are as above, then $\langle A, \alpha \rangle \in \mathbf{V}$ just in case $\langle B, \beta \rangle \in \mathbf{V}$).

The covariety $\mathcal{D}et$ is not closed under total bisimulations. Let $I = \{i\}$ and consider the automata $A = \{a\}$ and $B = \{b, c\}$ represented by the graphs below.

$$\begin{tikzcd} a \arrow[loop left, "i"] \end{tikzcd} \qquad \begin{tikzcd} b \arrow[loop left, "i"] \arrow[r, "i"] & c \arrow[loop right, "i"] \end{tikzcd}$$

These two automata are related by a total bisimulation, but $A$ is deterministic, while $B$ is not. Hence, $\mathcal{D}et$ cannot be defined by a coequation over 1.

**Example 3.11.** Fix a set $Z$ and let $\Gamma : \mathbf{Set} \to \mathbf{Set}$ be the functor

$$\Gamma X = Z \times X.$$

Any $\Gamma$-coalgebra $\langle A, \alpha \rangle$ can be viewed as a collection of streams over $Z$, then, in which the same stream may be multiply represented as elements of $A$.

The cofree coalgebra $H\mathbb{N}$ is the final $\mathbb{N} \times Z \times -$ coalgebra – i.e., $H\mathbb{N} = (\mathbb{N} \times Z)^{\omega}$. Given an element $\sigma \in H\mathbb{N}$, we can define

$$\mathcal{C}ol(\sigma) = \{\pi_1 \circ \sigma(i) \mid i < \omega\}$$

(equivalently, $\mathcal{C}ol(\sigma) = \{\varepsilon_{\mathbb{N}} \circ t^i(\sigma) \mid i < \omega\}$, where $t$ is the tail destructor). In other words, $\mathcal{C}ol(\sigma)$ is the set of all colors that occur in the stream $\sigma$. Define a coequation $\varphi$ over $\mathbb{N}$ by

$$\varphi = \{\sigma \in UH\mathbb{N} \mid \mathbf{card}(\mathcal{C}ol(\sigma)) < \aleph_0\},$$

so $\sigma \in \varphi$ just in case only finitely many colors occur in $\sigma$.

One can check that, for any $\Gamma$-coalgebra $\langle A, \alpha \rangle$, we have $\langle A, \alpha \rangle \models \varphi$ just in case, for all $a \in A$, there is $n \geq 0$, $m > 0$ such that

$$t^n(a) = t^{n+m}(a),$$

(where $\alpha = \langle h, t \rangle$). In other words, $\langle A, \alpha \rangle \models \varphi$ iff each stream in $A$ has only a finite number of "states".

**Remark 3.12.** If one is interested not in equality of states, but in the observable behavior of streams, then one might require instead that, for every $a \in A$, there is $n \geq 0$, $m > 0$ such that for all $i \geq 0$,

$$h \circ t^{n+i}(a) = h \circ t^{n+m+i}(a).$$

This condition can be specified by a coequation over 1 color, namely by the coequation

$$\{\sigma \in UH1 \mid \exists n, m \,.\, t^n(\sigma) = t^{n+m}(\sigma)\}.$$

**Remark 3.13.** One can easily generate other interesting coequations similar to Example 3.11. First, it's easy to see that the same idea can be used with polynomial functors in general. Second, one can require that each state begins repeating within $n$ applications of the destructors by replacing $\aleph_0$ with $n$ in the definition of $\varphi$.

## 4. Conditional coequations

In Definition 3.5, we introduced a coequation $\varphi$ over $C$ as a regular subobject

$$\varphi \rightarrowtriangle UHC$$

in $\mathcal{E}$. In this section, we generalize the notion of coequation to include regular subobjects

$$\varphi \rhd\!\!\longrightarrow \langle A,\, \alpha\rangle$$

where $\langle A,\, \alpha\rangle$ is an arbitrary coalgebra.

**Definition 4.1.** A *conditional coequation* over $\langle A,\, \alpha\rangle$ is any regular subobject $\varphi \leq A = U\langle A,\, \alpha\rangle$. We say that $\langle B,\, \beta\rangle \models_\alpha \varphi$ (or just $\langle B,\, \beta\rangle \models \varphi$) if and only if, for every homomorphism

$$p\colon\langle B,\, \beta\rangle \longrightarrow \langle A,\, \alpha\rangle\,,$$

$\mathsf{Im}(p) \leq \varphi$.

We sometimes drop the word "conditional" and refer to $\varphi \leq A$ as a *coequation over* $\langle A,\, \alpha\rangle$.

**Remark 4.2.** Let $C$ be a regular injective object in $\mathcal{E}$. Note that a coequation over $C$ in the sense of Definition 3.5 is the same as a conditional coequation over $HC$. There should be no confusion, as the objects of $\mathcal{E}$ are distinct from the objects of $\mathcal{E}_{\mathbb{G}}$.

We adopt the name "conditional coequation" because the semantics introduced in Definition 4.1 arise from the dual of conditional equations in the algebraic case. Given two coequations, $\varphi$ and $\psi$, over $C$, we say that $\langle B,\, \beta\rangle \models \varphi \Rightarrow \psi$ just in case, for every

$$p\colon\langle B,\, \beta\rangle \longrightarrow HC\,,$$

if $\langle B,\, \beta\rangle \models \varphi(p)$, then $\langle B,\, \beta\rangle \models \psi(p)$. (In (Kurz, 1999) and (Kurz, 2000), $\varphi \Rightarrow \psi$ is denoted $\varphi/\psi$.) See also (Roşu, pear) for a recent treatment of the dual notion and its relationship to equations.

Now, for any pair of coequations $\varphi$ and $\psi$ over $C$, there is a coalgebra $\langle A,\, \alpha\rangle$ and a conditional coequation $\vartheta$ over $\langle A,\, \alpha\rangle$ such that, for all $\langle B,\, \beta\rangle$,

$$\langle B,\, \beta\rangle \models \varphi \Rightarrow \psi \text{ iff } \langle B,\, \beta\rangle \models_\alpha \vartheta. \tag{1}$$

Namely, we take $\langle A,\, \alpha\rangle$ to be the largest subcoalgebra of $HC$ contained in $\varphi$, i.e., $\langle A,\, \alpha\rangle = [\varphi]_{HC}$ in the terminology of Section 6. We take $\vartheta = A \wedge \psi$ and show that (1) holds. On the other hand, given a conditional coequation $\vartheta$ over $\langle A,\, \alpha\rangle$, we can view both $\vartheta$ and $A$ as coequations over regular injective $C$, where $A \leq C$ — that is, as subobjects of $UHC$. It is easy to check that

$$\langle B,\, \beta\rangle \models_\alpha \vartheta \text{ iff } \langle B,\, \beta\rangle \models A \Rightarrow \vartheta.$$

**Remark 4.3.** Given coequations $\varphi$ and $\psi$ over $C$, one can consider the coequation $\varphi \to \psi$ over $C$, where $\to$ is the exponential in $\mathsf{Sub}(UHC)$ (assuming it exists). One can show that, if $\langle A,\, \alpha\rangle \models \varphi \to \psi$, then $\langle A,\, \alpha\rangle \models \varphi \Rightarrow \psi$, but the converse does not hold in general.

**Example 4.4.** Let $\Gamma- = - \times -$ and let $A = \{a, b\}$. Let

$$\langle \varepsilon_A,\, l,\, r\rangle\colon UHA \rhd\!\!\longrightarrow\!\!\rhd A \times UHA \times UHA$$

be the counit and structure map of $HA$. Define coequations $\varphi$ and $\psi$ over $A$ by

$$\varphi = \{\sigma \in UHA \mid \sigma = l(\sigma)\},$$
$$\psi = \{\sigma \in UHA \mid \sigma = r(\sigma)\}.$$

Let $\alpha(a) = \langle b, b\rangle$ and $\alpha(b) = \langle b, a\rangle$. Then $\langle A, \alpha\rangle \models \varphi \Rightarrow \psi$, but $\langle A, \alpha\rangle \not\models \varphi \to \psi$.

Conditional coequations provide a means of interpreting the quasi-covariety theorem, below. As before, we first state an abstract version of the quasi-variety theorem and then interpret the theorem in categories of coalgebras. The proof of Theorem 4.5 and its corollaries can be found in (Awodey and Hughes, 2000). The theorem also was proven independently by Alexander Kurz in (Kurz, 2000).

**Theorem 4.5.** Let $\mathcal{E}$ be a quasi-co-Birkhoff category and $\mathbf{V}$ a full subcategory of $\mathcal{E}$. The following are equivalent.

1 $\mathbf{V}$ is a quasi-covariety.
2 The inclusion $U^{\mathbf{V}}\!:\!\mathbf{V} \twoheadrightarrow \mathcal{E}$ has a right adjoint $H^{\mathbf{V}}$ such that each component of the counit $\varepsilon^{\mathbf{V}}\!:\!U^{\mathbf{V}}H^{\mathbf{V}} \Rightarrow 1_{\mathcal{E}}$ is a regular mono, i.e., $\mathbf{V}$ is a regular mono-coreflective subcategory of $\mathcal{E}$.
3 $\mathbf{V} = S\text{-}\mathbf{Proj}$ for some collection $S$ of regular monos.

**Corollary 4.6.** Let $\mathcal{E}$ be a quasi-co-Birkhoff category and $\mathbf{V}$ a quasi-covariety of $\mathcal{E}$ and let $H^{\mathbf{V}}$ be right adjoint to $U^{\mathbf{V}}\!:\!\mathbf{V} \twoheadrightarrow \mathcal{E}$, as in Theorem 4.5, (2). Then

1 The unit $\eta^{\mathbf{V}}\!:\!1_{\mathbf{V}} \Rightarrow H^{\mathbf{V}}U^{\mathbf{V}}$ is an isomorphism.
2 For each $C \in \mathcal{E}$, $C \in \mathbf{V}$ iff $C$ is $\{\varepsilon^{\mathbf{V}}_C\}\text{-}\mathbf{Proj}$, where $\varepsilon^{\mathbf{V}}$ is the counit of the adjunction $U^{\mathbf{V}} \dashv H^{\mathbf{V}}$.
3 The corresponding comonad, $\mathbb{G}^{\mathbf{V}} = \langle U^{\mathbf{V}}H^{\mathbf{V}}, \varepsilon, U^{\mathbf{V}}\eta_{H^{\mathbf{V}}}\rangle$, is idempotent.
4 The comonad $\mathbb{G}^{\mathbf{V}}$ preserves regular monos.

The following corollary restates the results of Theorem 4.5 for categories of coalgebras in terms of conditional coequations.

**Corollary 4.7.** Let $\mathcal{E}$ be quasi-co-Birkhoff and let $\Gamma\!:\!\mathcal{E} \twoheadrightarrow \mathcal{E}$ be a functor that preserves regular monos. A full subcategory $\mathbf{V}$ of $\mathcal{E}_\Gamma$ is a quasi-covariety just in case there is a collection $S$ of conditional coequations such that

$$\langle B, \beta\rangle \in \mathbf{V} \text{ iff } \forall\varphi \in S\ \langle B, \beta\rangle \models \varphi.$$

The same claim holds if we replace the endofunctor $\Gamma$ with a comonad $\mathbb{G}$.

The following corollary is a generalization of Theorem 12 from (Jacobs, 1995), where the author proves it for a restricted class of coequations over **Set**, namely those coequations that arise as equalizers of a pair of terms related to the functor $\mathbb{G}$.

**Corollary 4.8.** Let $\mathcal{E}$ be co-Birkhoff and $\mathbb{G}$ preserve regular monos, and let $\mathbf{V}$ be a covariety of $\mathcal{E}_{\mathbb{G}}$. Then the forgetful functor

$$\mathbf{V} \longrightarrow \mathcal{E}$$

is comonadic. Moreover, the associated comonad preserves regular monos and so $\mathbf{V}$ is again co-Birkhoff.

*Proof.* The forgetful functor $\mathbf{V} \to \mathcal{E}$ is the composite

$$\mathbf{V} \xrightarrow{U^{\mathbf{V}}} \mathcal{E}_{\mathbb{G}} \xrightarrow{U} \mathcal{E}\,.$$

To show that this composite is comonadic, it suffices to show (by the dual of Beck's theorem (Borceux, 1994, Volume 2, Theorem 4.4.4)) that the following hold:

1 $U \circ U^{\mathbf{V}}$ has a right adjoint;
2 $U \circ U^{\mathbf{V}}$ reflects isomorphisms;
3 $U \circ U^{\mathbf{V}}$ creates equalizers of pairs

$$\bullet \overset{f}{\underset{g}{\rightrightarrows}} \bullet$$

such that $U \circ U^{\mathbf{V}} f$, $U \circ U^{\mathbf{V}} g$ have a split equalizer in $\mathcal{E}$.

Condition (1) follows from Theorem 4.5. Condition (2) is easily verified and (3) follows from the same condition for $U$ and the fact that $U^{\mathbf{V}}$ creates equalizers. □

## 5. Deductive completeness and invariance

We focus now on Birkhoff's completeness theorem. Whereas the variety theorem gives an equivalence between closure conditions on classes of algebras and equationally defined classes, the completeness theorem states an equivalence between deductively closed sets of equations and theories for classes of algebras. We first recall the completeness theorem in the classical setting.

Let $\Sigma$ be a signature and $\mathbb{T}$ the associated monad (so that $\mathsf{Alg}(\Sigma) \cong \mathbf{Set}^{\mathbb{T}}$), and let

$$F : \mathbf{Set} \longrightarrow \mathbf{Set}^{\mathbb{T}}$$

be the left adjoint of the forgetful functor $U : \mathbf{Set}^{\mathbb{T}} \to \mathbf{Set}$. We say that a set of equations $E$ over $X$ (i.e., a subset of $UFX \times UFX$, by identifying $t_1 = t_2$ with the pair $\langle t_1, t_2 \rangle$) is *closed* if it satisfies the following:

(i) For each $x \in X$, $x = x \in E$;
(ii) For each $t_1 = t_2 \in E$, $t_2 = t_1 \in E$;
(iii) If $t_1 = t_2 \in E$ and $t_2 = t_3 \in E$, then $t_1 = t_3 \in E$;
(iv) For each function symbol $f^{(n)} \in \Sigma$, and each $n$-tuple of equations,

$$s_1 = t_1, \ldots, s_n = t_n,$$

in $E$, the equation $f^{(n)}(s_1, \ldots, s_n) = f^{(n)}(t_1, \ldots, t_n) \in E$.
(v) $E$ is closed under substitution of terms for variables. That is, for each $t_1 = t_2 \in E$, $t \in UFX$, $x \in X$,

$$t_1[t/x] = t_2[t/x] \in E.$$

The term $t_1[t/x]$ is defined as follows: Let $\sigma^t_x : X \twoheadrightarrow UFX$ be the map

$$\sigma^t_x(y) = \begin{cases} t & \text{if } x = y \\ y & \text{else} \end{cases}$$

Let $\widetilde{\sigma}^t_x : FX \twoheadrightarrow FX$ be the adjoint transpose of $\sigma^t_x$. Then $t_1[t/x] = \widetilde{\sigma}^t_x(t_1)$.

**Theorem 5.1 (Birkhoff's completeness theorem).** A set of equations $E$ is the equational theory for some class $\mathbf{V}$ of $\Sigma$-algebras just in case $E$ is closed. Equivalently, for any set $E$ of equations, the deductive closure of $E$ is the complete theory for the models of $E$, i.e.,

$$\mathsf{Ded}(E) = \mathsf{Th}\,\mathsf{Mod}(E).$$

We say that a (binary) relation $E$ over $UFX$ is *stable* just in case, for every homomorphism

$$f : FX \longrightarrow FX\,,$$

the image of $E$ under $f$ is contained in $E$, i.e.,

$$\exists_f E \leq E.$$

If $E$ is stable, then for every $x \in X$, $t \in UFX$, the image of $E$ under $\widetilde{\sigma}^t_x$ is contained in $E$ and so $E$ is closed under substitutions. Conversely, suppose $E$ is closed under substitutions. Let $f : FX \twoheadrightarrow FX$ be given, and $t_1 = t_2 \in E$, with $x_1, \ldots, x_n$ the free variables of $t_1$ and $t_2$. Then

$$t_1[f(x_1)/x_1][f(x_2)/x_2] \ldots [f(x_n)/x_n] = t_2[f(x_1)/x_1][f(x_2)/x_2] \ldots [f(x_n)/x_n] \in E,$$

and so $E$ is stable. Since (i) - (iv) hold iff $E$ is a congruence, we see that a set $E$ of equations over $X$ is closed just in case

(i′) $E$ is a congruence;
(ii′) $E$ is *stable*.

We can use the well-known isomorphism between congruences and quotients of $FX$ (in $\mathbf{Set}^{\mathbb{T}}$) to translate these conditions on sets of equations to a pair of conditions on quotients of $UFX$. Accordingly, one finds that a quotient $q : UFX \twoheadrightarrow Q$ is the coequalizer of a closed set $E$ of equations just in case

(i″) there is a structure map $\nu : TQ \twoheadrightarrow Q$ such that $q$ is a $\mathbb{T}$-homomorphism;
(ii″) for every endomorphism $f : FX \twoheadrightarrow FX$, there is a (necessarily unique)

$$g : \langle Q, \nu\rangle \longrightarrow \langle Q, \nu\rangle$$

such that $q \circ f = g \circ q$.

We dualize (ii″) to yield the notion of *endomorphism-invariant coequations* in the coalgebraic setting. This definition is first found in (Gumm and Schröder, 2001). The term endomorphism-invariant defined here should not be confused with the definition of an invariant predicate as one that admits a structure map (i.e., is the carrier of a subcoalgebra, also called a *mongruence* in (Jacobs, 1995)), as used in (Jacobs, 1999; Mašulović, 2001;

Poll and Zwanenburg, 2001) and elsewhere. Nonetheless, in what follows, we use "invariant" as a shorthand term for "endomorphism-invariant" and hope that no confusion will result.

**Definition 5.2.** Let $\langle A, \alpha\rangle$ be a $\mathbb{G}$-coalgebra. We say that a regular subobject $\varphi$ of $A$ is *endomorphism-invariant* (hereafter, *invariant*) just in case, for every homomorphism

$$p : \langle A, \alpha\rangle \longrightarrow \langle A, \alpha\rangle\,,$$

the image of $\varphi$ under $p$ is contained in $\varphi$, i.e.,

$$\exists_p \varphi \leq \varphi.$$

**Remark 5.3.** Notice that an invariant coequation arises as the formal dual of a stable set of equations. We do not offer here a set of deductive rules with which one can reason about coalgebras – that is, we do not offer a deductive analogue to Birkhoff's completeness theorem. See, however, (Corradini, 1998) for an equational calculus which is complete with respect to certain classes of covarieties and also (Goldblatt, 2001b; Goldblatt, 2001a) for another equational logic intended to provide an analogue to Birkhoff's completeness theorem, rather than a formal dual. Furthermore, we have the first steps for a complete deductive calculus for coequations (and also one for conditional coequations), using the results proven here. We hope to present these results in a subsequent paper.

**Remark 5.4.** If $\langle A, \alpha\rangle$ is a subcoalgebra of the final coalgebra, then any conditional coequation $\varphi$ over $\langle A, \alpha\rangle$ is endomorphism-invariant, because the identity is the only endomorphism.

Given a coequational variety

$$\mathbf{V} = \{\langle B, \beta\rangle \mid \langle B, \beta\rangle \models \psi\},$$

we are interested in the minimal (intuitively, strongest) coequation $\varphi$ such that $\mathbf{V} \models \varphi$. Such minimal coequations can be viewed as generating the collection of coequations that $\mathbf{V}$ satisfies, in the sense that, for any coequation $\vartheta$, if $\mathbf{V} \models \vartheta$, then $\varphi \vdash \vartheta$. In this sense, the minimal coequation represents the coequational theory of $\mathbf{V}$ — it represents the coequational commitment that $\mathbf{V}$ entails. This intuition motivates the following definition.

**Definition 5.5.** Let $\varphi$ be a (conditional) coequation over $\langle A, \alpha\rangle$ and $\mathbf{V}$ a collection of coalgebras. We say that $\varphi$ is the *generating (conditional) coequation* for $\mathbf{V}$ just in case

1 $\mathbf{V} \models \varphi$;
2 For any conditional coequation $\psi$ over $\langle A, \alpha\rangle$, if $\mathbf{V} \models \psi$ then $\varphi \vdash \psi$.

Now we are in a position to state our main result.

**Theorem 5.6 (Invariance theorem).** A coequation $\varphi$ over $C$ is the generating coequation for some collection $\mathbf{V}$ of coalgebras just in case $\varphi$ is an invariant subcoalgebra of $HC$.

We postpone the proof until we've defined the modal operators $\Box$ and $\boxslash$. The invariance theorem first arises in (Gumm and Schröder, 2001), where it is proved for coalgebras over **Set**. The theorem is stated in different terms in their work, since it is not motivated by the coequation-as-predicate view that we take here.

## 6. The subcoalgebra operator

In the next two sections, we construct the modal operators that are used in the proof of the invariance theorem, and prove some basic results regarding these operators. Throughout what follows, we assume that $\mathcal{E}$ is co-Birkhoff and has pullbacks and that $\mathbb{G}$ preserves regular monos and pullbacks of regular monos, so that $\mathcal{E}_{\mathbb{G}}$ is co-Birkhoff and $U$ creates pullbacks of regular monos (and, in particular, finite intersections). We further assume that, for each $A \in \mathcal{E}$, $\mathsf{Sub}(A)$ is a Heyting algebra. This last assumption is used only so that one can introduce the usual axioms for **S4** modal operators. It is not necessary for the invariance theorem.

In this section, we introduce the modal operator $\Box$. Given a subobject $\varphi$ of $A = U\langle A, \alpha\rangle$, $\Box\varphi$ is the carrier of the greatest subcoalgebra of $A$ contained in $\varphi$. The construction is well-known, although the view that $\Box$ is a modal operator is perhaps less familiar. The $\Box$ operator is discussed in (Jacobs, 1999), where it plays a central role. It is from that work that we take the view of $\Box$ as a "henceforth" operator.

Since the coalgebraic forgetful functor $U : \mathcal{E}_{\mathbb{G}} \to \mathcal{E}$ preserves regular monos, there is an induced forgetful functor,

$$U_\alpha : \mathsf{Sub}(\langle A, \alpha\rangle) \longrightarrow \mathsf{Sub}(A),$$

from the partial order of regular subobjects of $\langle A, \alpha\rangle$ to the partial order of regular subobjects of $A$. As is well known, $U_\alpha$ has a right adjoint, which we denote $[-]_\alpha$ (dropping the subscripts whenever convenient). The right adjoint maps a subobject $B \leq A$ to the largest subcoalgebra contained in $B$. More precisely,

$$[B] = \bigvee \{\langle C, \gamma\rangle \leq \langle A, \alpha\rangle \mid C \leq B\}.$$

Here, we use the fact that $U_\alpha$ creates joins. Alternatively, one may define $[B]$ as the pullback shown below, where $U \dashv H$.

$$\begin{tikzcd} {[B]} \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & HB \arrow[d, tail] \\ \langle A, \alpha\rangle \arrow[r, tail] & HA \end{tikzcd}$$

This adjoint pair yields a modal operator

$$\Box_\alpha : \mathsf{Sub}(A) \longrightarrow \mathsf{Sub}(A),$$

as usual, by taking the composite $\Box_\alpha = U_\alpha \circ [-]_\alpha$. Again, we drop the subscript when convenient.

The following two theorems were first presented in (Jacobs, 1999).

**Theorem 6.1.** $\Box$ is an **S4** necessity operator; i.e., it satisfies the following:

1 If $\varphi \vdash \psi$, then $\Box\varphi \vdash \Box\psi$

2 $\Box\varphi \vdash \varphi$
3 $\Box\varphi \vdash \Box\,\Box\,\varphi$
4 $\Box(\varphi \to \psi) \vdash \Box\varphi \to \Box\psi$

*Proof.* Condition (1) is just functoriality, and conditions (2) and (3) are just the counit and comultiplication for the comonad $\Box$.

The last item follows from the fact that $U_\alpha$ preserves meets, and hence so does $\Box$. The argument for (4) from this is standard, but we include it here.

By (1), we have

$$\Box((\varphi \to \psi) \wedge \varphi) \vdash \Box\psi,$$

and, hence,

$$\Box(\varphi \to \psi) \wedge \Box\varphi \vdash \Box\psi.$$

Therefore, $\Box(\varphi \to \psi) \vdash \Box\varphi \to \Box\psi$. $\square$

**Theorem 6.2.** $\Box$ is stable under pullback along homomorphisms. That is, for any

$$f : \langle A,\, \alpha\rangle \longrightarrow \langle B,\, \beta\rangle\,,$$

we have

$$\Box_\alpha \circ f^* = f^* \circ \Box_\beta.$$

*Proof.* Consider the diagram below.

$$\begin{tikzcd}
\Box f^*P \arrow[rr, tail] \arrow[dr] \arrow[dd, tail] & & Gf^*P \arrow[dr] \arrow[dd, tail] & \\
& \Box P \arrow[rr, tail] \arrow[dd, tail] & & GP \arrow[dd, tail] \\
A \arrow[rr, tail] \arrow[dr] & & GA \arrow[dr] & \\
& B \arrow[rr, tail] & & GB
\end{tikzcd}$$

The bottom face commutes, since $f$ is a homomorphism. The front and rear faces are pullbacks by definition of $\Box$, and the right face is a pullback since $G$ preserves pullbacks along regular monos by assumption. Hence, the left face is a pullback. $\square$

Theorem 6.2 can be understood as a statement about substitution of terms for variables. Namely, we view conditional coequations $\varphi$ over $\langle A,\, \alpha\rangle$ as predicates of a single variable $x$ of type $A$. Then, Theorem 6.2 says that, for any homomorphism

$$f : \langle B,\, \beta\rangle \longrightarrow \langle A,\, \alpha\rangle\,,$$

and any variable $y$ of type $B$, we have

$$(\Box\varphi)[f(y)/x] = \Box(\varphi[f(y)/x]).$$

Thus, $\Box$ is stable under substitutions of terms built from homomorphisms for variables. (It is not stable under substitution of arbitrary terms for variables, however.)

## 7. The invariance operator

We apply the same approach to invariant coequations as in Section 6. That is, we first define an adjoint pair (a Galois correspondence) between the coequations over $\langle A, \alpha\rangle$ and the invariant coequations. Then, we use this pair to define a modal operator $\boxslash$ on coequations over $\langle A, \alpha\rangle$. Again, we assume that $\mathsf{Sub}(UHC)$ is a Heyting algebra, so that we may use the familiar axioms for **S4** necessity operators.

Accordingly, let $\mathsf{Inv}(\langle A, \alpha\rangle)$ denote the full subcategory of $\mathsf{Sub}(A)$ consisting of the invariant coequations over $\langle A, \alpha\rangle$, and let

$$I_\alpha : \mathsf{Inv}(\langle A, \alpha\rangle) \longrightarrow \mathsf{Sub}(A)$$

be the inclusion functor.

**Theorem 7.1.** $I_\alpha$ has a right adjoint.

*Proof.* Let $\varphi \leq A$ and define

$$\mathfrak{P}_\varphi = \{\psi \leq A \mid \forall p : \langle A, \alpha\rangle \longrightarrow \langle A, \alpha\rangle\, (\exists_p \psi \leq \varphi)\}.$$

We define a functor $J_\alpha : \mathsf{Sub}(A) \to \mathsf{Sub}(A)$ by

$$J_\alpha(\varphi) = \bigvee \mathfrak{P}_\varphi,$$

omitting the subscripts when convenient.

We first show that $J\varphi$ is invariant. Let

$$r : \langle A, \alpha\rangle \longrightarrow \langle A, \alpha\rangle$$

be given. In order to show that $\exists_r J\varphi \leq J\varphi$, it suffices to show that $\exists_r J\varphi \in \mathfrak{P}_\varphi$, i.e., for every homomorphism $p : \langle A, \alpha\rangle \to \langle A, \alpha\rangle$, we have $\exists_p(\exists_r J\varphi) \leq \varphi$. A quick calculation shows

$$\exists_p \exists_r J\varphi = \exists_{p \circ r} \bigvee \mathfrak{P}_\varphi = \bigvee \{\exists_{p \circ r} \psi \mid \psi \in \mathfrak{P}_\varphi\} \leq \varphi.$$

Next, we show that $I \dashv J$. Let $\psi$ be invariant. If $\psi \leq \varphi$, then, for every endomorphism $p$,

$$\exists_p \psi \leq \psi \leq \varphi,$$

so $\psi \in \mathfrak{P}_\varphi$ and hence $\psi \leq J\varphi$. On the other hand, if $\psi \leq J\varphi$, then

$$\psi \leq J\varphi \leq \varphi.$$

$\Box$

Now let $\boxslash_\alpha = I_\alpha J_\alpha$. In terms of the elements $a \in A = U\langle A, \alpha\rangle$, we see that $a \in \boxslash_\alpha \varphi$ if and only if for every homomorphism $p : \langle A, \alpha\rangle \to \langle A, \alpha\rangle$, we have $p(a) \in \varphi$. Indeed, one may also show that $a \in \boxslash \varphi$ if and only if for every homomorphism $p : \langle A, \alpha\rangle \to \langle A, \alpha\rangle$, we have $p(a) \in \boxslash \varphi$.

**Theorem 7.2.** $\boxslash$ is an **S4** necessity operator.

*Proof.* Again, since $\boxslash$ is a comonad, it suffices to show that $\boxslash$ preserves meets, or, more specifically, that

$$\boxslash\varphi \wedge \boxslash\psi \vdash \boxslash(\varphi \wedge \psi).$$

Let $p\colon\langle A,\, \alpha\rangle \twoheadrightarrow \langle A,\, \alpha\rangle$ be given (where $\varphi$ and $\psi$ are conditional coequations over $\langle A,\, \alpha\rangle$). Then

$$\exists_p(\boxslash\varphi \wedge \boxslash\psi) \leq \exists_p \boxslash \varphi \leq \varphi$$

and, similarly, $\exists_p(\boxslash\varphi \wedge \boxslash\psi) \leq \psi$. Hence, $\exists_p(\boxslash\varphi \wedge \boxslash\psi) \leq \varphi \wedge \psi$. Since $p$ was an arbitrary homomorphism, $\boxslash\varphi \wedge \boxslash\psi \vdash \boxslash(\varphi \wedge \psi)$. □

**Remark 7.3.** If $\langle A,\, \alpha\rangle$ is a subcoalgebra of the final coalgebra, then every conditional coequation over $\langle A,\, \alpha\rangle$ is invariant (see Remark 5.4). Hence, in this case, $\boxslash_\alpha$ is just the identity functor $\mathsf{Sub}(A) \twoheadrightarrow \mathsf{Sub}(A)$.

**Remark 7.4.** Unlike $\square$, the operator $\boxslash$ does not commute with pullbacks along homomorphisms. Indeed, let $\Gamma\colon\mathbf{Set} \twoheadrightarrow \mathbf{Set}$ be the identity functor. We will consider a coequation $\varphi$ over 2 colors, that is, a subset of $UH2 = 2^\omega$, the set of streams over 2. Specifically, let

$$\varphi = \{\underline{0},\, \underline{1}\},$$

where $\underline{0}$ and $\underline{1}$ are the constant streams. Note that $\varphi$ is invariant.

Let $p\colon H3 \twoheadrightarrow H2$ be the homomorphism induced by the coloring $\overline{p}\colon 3 \twoheadrightarrow 2$, where

$$\overline{p}(0) = 0, \qquad \overline{p}(1) = 0, \qquad \overline{p}(2) = 1$$

(i.e., $p = H(\overline{p})$). Then $p^*\varphi$ is the set

$$\{\sigma \in 3^\omega \mid \forall n\ \sigma(n) < 2\} \cup \{\underline{2}\}.$$

It is easy to check that

$$\boxslash p^*\varphi = \{\underline{0},\, \underline{1},\, \underline{2}\} \neq p^*(\boxslash\varphi) = p^*\varphi.$$

In terms of substitutions, then, it is not the case that, for every homomorphism

$$f\colon\langle B,\, \beta\rangle \longrightarrow \langle A,\, \alpha\rangle,$$

$(\boxslash\varphi)[f(y)/x] = \boxslash(\varphi[f(y)/x])$.

We return to the examples of Section 3 to give some idea of how $\boxslash$ works. In those examples, the coequations over $C$ were described in terms of the coloring $\varepsilon_C$. Typically, $\boxslash$ takes a coequation defined in terms of colorings to a similar coequation defined in terms of equality of states, as these examples illustrate.

**Example 7.5.** Let $\Gamma S = (\mathcal{P}_{\mathsf{fin}} S)^{\mathcal{I}}$, as in Example 3.10. Recall that the class of deterministic automata $\mathcal{D}et$ forms a covariety of $\mathbf{Set}_\Gamma$, where the defining coequation $\varphi$ over 2 is given by

$$\varphi = \{x \in UH2 \mid \forall i \in \mathcal{I}\, \forall y, z \in \sigma(x)(i) \,.\, \varepsilon_2(y) = \varepsilon_2(z)\}.$$

It is easy to show that

$$\boxslash\varphi = \{x \in UH2 \mid \forall i \in \mathcal{I}\, \forall y, z \in \sigma(x)(i) \,.\, y = z\},$$

or, more simply,

$$\boxslash\varphi = \{x \in UH2 \mid \forall i \in \mathcal{I} \,.\, \mathbf{card}(\sigma(x)(i)) < 2\}.$$

Indeed, let $\psi \leq UH2$ and suppose that for all endomorphisms $p\colon H2 \twoheadrightarrow H2$, we have $\exists_p \psi \leq \varphi$. Suppose, for sake of contradiction, that there is an $x \in \psi$, $i \in I$ such that $\mathbf{card}(\sigma(x)(i)) \geq 2$. Let $y$ and $z$ be distinct elements of $\sigma(x)(i)$ and define $c\colon UH2 \twoheadrightarrow 2$ taking $y$ to 0 and every other element of $UH2$ to 1. Let $\widetilde{c}$ be the adjoint transpose of $c$ and we see that $\varepsilon_2(\widetilde{c}(y)) \neq \varepsilon_2(\widetilde{c}(z))$. This implies that $\widetilde{c}(x) \notin \varphi$, contradicting our assumption that for every endomorphism $p$, $\exists_p \psi \leq \varphi$.

**Example 7.6.** Recall the functor $\Gamma X = Z \times X$ and the coequation $\varphi$ over $\mathbb{N}$ defined by

$$\varphi = \{\sigma \in UH\mathbb{N} \mid \mathbf{card}(\mathcal{C}ol(\sigma)) < \aleph_0\},$$

from Example 3.11. For each $\sigma \in UH\mathbb{N}$, let

$$\mathcal{S}t(\sigma) = \{t^n(\sigma) \mid n \in \omega\},$$

where $\langle \varepsilon_{\mathbb{N}}, h, t\rangle \colon UH\mathbb{N} \rightarrowtriangle \mathbb{N} \times Z \times UH\mathbb{N}$ is the counit and structure map for $H\mathbb{N}$. Then

$$\boxslash\varphi = \{\sigma \in UH\mathbb{N} \mid \mathbf{card}(\mathcal{S}t(\sigma)) < \aleph_0\}.$$

## 8. Generating coequations

We return to the proof of the invariance theorem. To begin, we show that, for any $\varphi$ over $\langle A, \alpha\rangle$, $\boxslash\varphi$ and $\Box\varphi$ have the same expressive power as $\varphi$ – i.e., define the same quasi-covariety.

**Theorem 8.1.** Let $\langle A, \alpha\rangle$ be given. For every $\varphi \in \mathsf{Sub}(A)$, $\langle B, \beta\rangle \in \mathcal{E}_{\mathbb{G}}$, the following are equivalent.

1 $\langle B, \beta\rangle \models \varphi$
2 $\langle B, \beta\rangle \models \boxslash\varphi$
3 $\langle B, \beta\rangle \models \Box\varphi$

*Proof.* We begin with (1) $\Leftrightarrow$ (2). Since $\boxslash\varphi \vdash \varphi$, trivially (2) $\Rightarrow$ (1). Suppose, then, that $\langle B, \beta\rangle \models \varphi$. Let $p\colon \langle B, \beta\rangle \twoheadrightarrow \langle A, \alpha\rangle$ be given. To show that $\mathsf{Im}(p) \leq \boxslash\varphi$, we will show that, for every $r\colon \langle A, \alpha\rangle \twoheadrightarrow \langle A, \alpha\rangle$, $\exists_r \mathsf{Im}(p) \leq \varphi$. But, $\exists_r \mathsf{Im}(p) = \mathsf{Im}(r \circ p) \leq \varphi$, since $\langle B, \beta\rangle \models \varphi$.

For the equivalence (3) $\Leftrightarrow$ (1), again we note that one direction, namely (3) $\Rightarrow$ (1), is trivial. Let $\langle B, \beta\rangle \models \varphi$ and let $p\colon \langle B, \beta\rangle \twoheadrightarrow \langle A, \alpha\rangle$ be given. Then $U_\alpha \mathsf{Im}(p) = \mathsf{Im}(Up) \leq \varphi$ and so, by the adjunction $U_\alpha \dashv [-]_\alpha$, $\mathsf{Im}(p) \leq [\varphi]_\alpha$. Thus,

$$\mathsf{Im}(Up) = U_\alpha \mathsf{Im}(p) \leq U_\alpha[\varphi]_\alpha = \Box_\alpha\varphi.$$

$\Box$

**Lemma 8.2.** Let $\varphi$ be a coequation over regular injective $C$. Then the coalgebra $[\boxslash\varphi]$ satisfies the coequation $\varphi$.

*Proof.* Let $p\colon[\boxslash\varphi]\rightarrow HC$ be given. Because $HC$ is regular injective, $p$ extends to a homomorphism $HC\rightarrow HC$, as shown below.

$$\begin{tikzcd}
UHC \arrow[r, dotted] & UHC \\
\Box \boxslash \varphi \arrow[u, tail] \arrow[ur, "p"] \arrow[r, dotted] & \boxslash\varphi \arrow[u, tail]
\end{tikzcd}$$

Hence, because $\Box \boxslash \varphi \vdash \boxslash\varphi$ and $\boxslash\varphi$ is invariant, there is a unique map $\Box \boxslash \varphi \rightarrow \boxslash \varphi$ making the square and thus the lower triangle commute, as desired. $\Box$

We say that a coequation $\varphi$ over $C$ is an *(endomorphism-)invariant subcoalgebra* just in case $\varphi = \Box \boxslash \varphi$. Clearly, if $\varphi = \Box\varphi = \boxslash\varphi$, then $\varphi$ is an invariant subcoalgebra. As we will see, the converse is true as well.

**Theorem 8.3 (Invariance theorem).** A coequation $\varphi$ over $C$ is the generating coequation for some collection $\mathbf{V}$ of coalgebras just in case $\varphi$ is an invariant subcoalgebra of $HC$.

*Proof.* If $\varphi$ is a generating coequation for some $\mathbf{V}$, then $\varphi \vdash \Box \boxslash \varphi$ by Theorem 8.1.

Conversely, suppose $\varphi = \Box \boxslash \varphi$ and define

$$\mathbf{V} = \{\langle B, \beta\rangle \mid \langle B, \beta\rangle \models \varphi\}.$$

Then, clearly, $\mathbf{V} \models \varphi$. We will show that, if $\mathbf{V} \models \psi$, then $\varphi \vdash \psi$. But, from Lemma 8.2, we know that $[\boxslash\varphi] = [\varphi]$ is in $\mathbf{V}$. Consequently, the inclusion $\Box\varphi \rightarrowtail UHC$ factors through $\psi$, and hence (using the fact that $\Box\varphi = \varphi$), we see that $\varphi \vdash \psi$. $\Box$

**Remark 8.4.** The same claim and proof holds for conditional coequations over $\langle A, \alpha\rangle$ where $\langle A, \alpha\rangle$ is regular injective or $\langle A, \alpha\rangle$ is an invariant subcoalgebra of $HA$. That is, a conditional coequation $\varphi$ over such $\langle A, \alpha\rangle$ is a generating coequation for some class $\mathbf{V}$ just in case $\varphi = \Box \boxslash \varphi$.

**Remark 8.5.** If $\mathcal{E}$ has arbitrary intersections of regular subobjects, we may re-state Theorem 8.3 as follows. Given a set $S$ of coequations over $C$, define

$$\mathsf{Mod}(S) = \{\langle A, \alpha\rangle \mid \langle A, \alpha\rangle \models S\}.$$

Also, given a collection $\mathbf{V}$ of coalgebras, define

$$\mathsf{Gen}(\mathbf{V}) = \bigwedge\{\psi \leq UHC \mid \mathbf{V} \models \psi\}.$$

Then,

$$\Box \boxslash \bigwedge S = \mathsf{Gen}\,\mathsf{Mod}(S).$$

Compare this to the statement of the completeness theorem in the introduction, namely,

$$\mathsf{Ded}(S) = \mathsf{Th}\,\mathsf{Mod}(S).$$

**Remark 8.6.** Let $\varphi$ be a coequation over $C$ and $\mathbf{V}_\varphi$ the covariety it defines. By Corollary 4.8, we know that the inclusion functor $U^\varphi : \mathbf{V}_\varphi \to \mathcal{E}_\mathbb{G}$ has a right adjoint $H^\varphi$. Let $\varepsilon^\varphi$ be the counit of the adjunction. The coalgebra $H^\varphi HC$ is *cofree for* $\mathbf{V}$ *over* $C$ in the following sense:

1 $H^\varphi HC \in \mathbf{V}$;
2 For any $\langle A, \alpha\rangle \in \mathbf{V}$, and any coloring $p : A \to C$, there is a unique homomorphism $\widetilde{p} : \langle A, \alpha\rangle \to H^\varphi HC$ such that $\varepsilon_C \circ U\varepsilon^\varphi_{HC} \circ \widetilde{p} = p$.

One can show that $UU^\varphi H^\varphi HC = \Box \boxslash \varphi$. In other words, $\Box \boxslash \varphi$ is the carrier of the cofree for $\mathbf{V}$ over $C$ coalgebra.

**Remark 8.7.** Suppose that the monad $\mathbb{G} : \mathcal{E} \to \mathcal{E}$ is bounded by $C$, in the sense of (Gumm and Schröder, 2001), so that, by Remark 3.9, for each covariety $\mathbf{V}$ of $\mathcal{E}_\mathbb{G}$, there is a coequation $\varphi \leq UHC$ such that $\mathbf{V} = \{\langle B, \beta\rangle \mid \langle B, \beta\rangle \models \varphi\}$. Let $\mathcal{P}(\mathcal{E}_\mathbb{G})$ denote the (large) partial order of subclasses of $\mathcal{E}_\mathbb{G}$. Then, there is a Galois connection

$$\mathcal{P}(\mathcal{E}_\mathbb{G}) \underset{\mathsf{Mod}}{\overset{\mathsf{Gen}}{\rightleftarrows}} \mathsf{Sub}_\mathcal{E}(UHC).$$

This adjunction yields an isomorphism between the fixed points of the compositions, namely,

$$\mathsf{CoVar}(\mathcal{E}_\mathbb{G}) \cong \mathsf{InvSub}(UHC),$$

where $\mathsf{CoVar}(\mathcal{E}_\mathbb{G})$ is the partial order of covarieties of $\mathcal{E}_\mathbb{G}$, and $\mathsf{InvSub}(UHC)$ the partial order of invariant subcoalgebras.

**Example 8.8.** Consider again the functor $\Gamma : \mathbf{Set} \to \mathbf{Set}$ where $\Gamma S = (\mathcal{P}_{\mathsf{fin}} S)^\mathcal{I}$ and the coequation $\varphi$ defined by

$$\varphi = \{x \in UH2 \mid \forall i \in \mathcal{I}\, \forall y, z \in \sigma(x)(i)\,.\, \varepsilon_2(y) = \varepsilon_2(z)\}.$$

We showed in Example 7.5 that

$$\boxslash\varphi = \{x \in UH2 \mid \forall i \in \mathcal{I}\ \mathbf{card}(\sigma(x)(i)) < 2\}.$$

We write $s \to s'$ if there is an $i$ such that $s \xrightarrow{i} s'$ and we write $\xrightarrow{*}$ for the transitive closure of $\to$. One can further show that

$$\Box \boxslash \varphi = \{x \in UH2 \mid \forall w \in UH2 (\text{if } x \xrightarrow{\ \ *\ \ } w \text{ then } \forall i \in \mathcal{I}\,.\,\mathbf{card}(\sigma(w)(i)) < 2)\}.$$

By Theorem 8.3, $\Box \boxslash \varphi$ is the generating coequation for $\mathcal{D}et$, the class of deterministic automata.

**Theorem 8.9.** For any coalgebra $\langle A, \alpha\rangle$,

$$\Box_\alpha \boxslash_\alpha \leq \boxslash_\alpha \Box_\alpha\,.$$

*Proof.* By definition of $\boxslash$, it suffices to show that, for every coequation $\varphi \leq A$ and homomorphism $p : \langle A, \alpha\rangle \to \langle A, \alpha\rangle$, $\exists_p \Box \boxslash \varphi \leq \Box\varphi$. Let $i : [\boxslash\varphi] \rightarrowtail HC$ denote the inclusion homomorphism. We calculate

$$\exists_p \Box \boxslash\varphi = \mathsf{Im}(p \circ i) \leq \Box\varphi,$$

since $[\boxslash\varphi] \models \varphi$ by Lemma 8.2 and hence $[\boxslash\varphi] \models \Box\varphi$ by Theorem 8.1. $\Box$

**Corollary 8.10.** The composite $\Box\boxslash$ is an **S4** operator.

*Proof.* It suffices to check that $\Box\boxslash$ is idempotent, since the other conditions are trivially satisfied. Idempotence follows from the following calculation.

$$\Box\boxslash = \Box\;\Box\;\boxslash\boxslash \leq \Box\;\boxslash\;\Box\boxslash$$

$\Box$

Hence, if $\varphi$ is an invariant subcoalgebra, then $\boxslash\varphi = \boxslash\;\Box\;\boxslash\varphi \geq \Box\;\boxslash\;\boxslash\varphi = \varphi$ and similarly $\Box\varphi = \varphi$. That is, $\varphi = \Box\;\boxslash\;\varphi$ iff $\varphi = \Box\varphi = \boxslash\varphi$.

We can prove that $\Box$ commutes with $\boxslash$ given further assumptions. Namely, if the modal operator $\Box$ has a left adjoint $\lhd$, then $\Box\boxslash = \boxslash\Box$. If the comonad $\mathbb{G}$ preserves non-empty intersections, then there is indeed such an adjoint $\lhd\dashv\Box$. In this case, the subcoalgebra forgetful functor $U_\alpha$ has a left adjoint,

$$F_\alpha : \mathsf{Sub}(A) \longrightarrow \mathsf{Sub}(\langle A,\, \alpha\rangle)\,,$$

taking a subobject $\varphi$ to the least subcoalgebra $\langle B,\, \beta\rangle$ such that $\varphi \leq B$. The closure operator $\lhd_\alpha$ is the composite $U_\alpha F_\alpha$.

See (Gumm, 2001c) for a discussion of functors which preserve non-empty intersections. Gumm also shows that the filter functor $\mathcal{F} : \mathbf{Set} \twoheadrightarrow \mathbf{Set}$ taking a set $X$ to the collection of filters on $X$ does not have this property. See also (Jacobs, 1999) for a discussion of the closure operator $\lhd_\alpha$, where it is denoted $\underset{\Leftarrow}{\alpha}$ (and $\Box$ is denoted $\underset{\Rightarrow}{\alpha}$). There, the author motivates the operators as temporal operators, with $\underset{\Rightarrow}{\alpha}$ acting as a "henceforth" operator and $\underset{\Leftarrow}{\alpha}$ a "sometime earlier" operator.

**Theorem 8.11.** If $\Box_\alpha$ has a left adjoint, $\lhd_\alpha$, then $\boxslash_\alpha\Box_\alpha = \Box_\alpha\boxslash_\alpha$.

*Proof.* To show that $\boxslash\Box \leq \Box\boxslash$, it is sufficient (by the adjunction $\lhd\dashv\Box$) to show that $\lhd\;\boxslash\Box \leq \boxslash$.

Let $\varphi \leq A = U\langle A,\, \alpha\rangle$. We will show that, for every homomorphism $p : \langle A,\, \alpha\rangle \twoheadrightarrow \langle A,\, \alpha\rangle$, $\exists_p \lhd\;\boxslash\;\Box\;\varphi \leq \varphi$ and conclude (by definition of $\boxslash$) that $\lhd\;\boxslash\;\Box\;\varphi \leq \boxslash\varphi$. Again, by the adjunctions, it suffices to show that

$$\boxslash\;\Box\;\varphi \leq \Box p^*\varphi = p^*\;\Box\;\varphi,$$

or, equivalently, $\exists_p\;\boxslash\;\Box\varphi \leq \Box\varphi$. This is immediate from the definition of $\boxslash$. $\Box$

**Example 8.12.** Let $\Gamma X = Z \times X$ and consider again the coequation $\varphi$ over $\mathbb{N}$ from Example 3.11, where

$$\varphi = \{\sigma \in UH\mathbb{N} \mid \mathbf{card}(\mathcal{C}ol(\sigma)) < \aleph_0\}.$$

Note that $\Gamma$ preserves non-empty intersections, so there is indeed an adjoint $\lhd\dashv\Box$ and hence Theorem 8.11 applies. It is easy to check that $\Box\varphi = \varphi$, and so $\boxslash\;\Box\;\varphi = \boxslash\varphi$ (which was described in Example 7.6) is the generating coequation for the covariety $\mathbf{V}_\varphi$.

**Example 8.13.** Consider the real interval $X = (0, 1]$, topologized with open sets of the form $(x, 1]$ for $x \in X$. Let $\mathcal{F}: \mathbf{Set} \twoheadrightarrow \mathbf{Set}$ be the filter functor and let $\xi: X \twoheadrightarrow \mathcal{F}X$ be the neighborhood filter map, as in (Gumm, 2001c). Recall from ibid that a map $X \twoheadrightarrow X$ is an $\mathcal{F}$-homomorphism just in case it is continuous and open. We will construct a coequation $\varphi$ over $\langle X, \xi \rangle$ such that $\Box \boxslash \varphi \neq \boxslash \Box \varphi$.

First, we note that the coequation $\{1\}$ is invariant, i.e., any continuous and open map $X \twoheadrightarrow X$ fixes 1. This is not difficult to show. Second, $\{1\}$ is the only non-trivial invariant subset of $X$. Indeed, if $x < 1$ is an element of $\varphi$ and $y \notin \varphi$, then it is easy to see that the map $f(z) = \frac{y}{x} z$ is a continuous, open map such that $\exists_f \varphi \not\leq \varphi$.

Let $\varphi = (\frac{1}{2}, 1]$, so that $\Box \varphi = \varphi$ and hence $\boxslash \Box \varphi = \{1\}$. However, $\Box \boxslash \varphi = \Box\{1\} = \emptyset$, since $\{1\}$ is not open. Indeed, the only generating conditional coequations over $\langle X, \xi \rangle$ are the trivial coequations, $X$ and $\emptyset$.

## 9. Future research

We have tried to develop the idea of "coequation-as-predicate" here. This approach naturally gives a means of constructing new coequations out of old, by using the standard logical operators $\wedge$, $\neg$, $\exists$, etc., as well as the modal operators $\Box$ and $\boxslash$. We have shown that, for any coequation $\varphi$, the covariety $\varphi$ defines is just the same covariety that $\Box\varphi$ and $\boxslash\varphi$ defines. It is also obvious that the covariety $\varphi \wedge \psi$ defines is the intersection of the covarieties defined by $\varphi$ and $\psi$. One would like to investigate the relation between the other logical operators (especially the quantifiers) and the partial order of covarieties.

The basic approach to Birkhoff's variety theorem found here first occurs in (Banaschewski and Herrlich, 1976) and was extended in a number of papers by Andréka and Németi, including (Andréka and Németi, 1978; Andréka and Németi, 1979a; Andréka and Németi, 1979b; Andréka and Németi, 1981; Németi and Sain, 1981; Németi, 1982; Andréka and Németi, 1983). In these papers, the authors give a sophisticated account of satisfaction-as-injectivity, allowing a characterization of classes of algebras defined by Horn equations, quasi-equations, etc. It would be worthwhile to take this work and explicitly dualize it, to see what the dual of the various equational classes are. We have recently spent some time exploring the dual of (Németi and Sain, 1981) in detail and will discuss these results in a forthcoming paper.

Robert Goldblatt has recently suggested an alternative approach to a coalgebraic analogue (not formal dual) of Birkhoff's variety theorem in (Goldblatt, 2001b; Goldblatt, 2001a). Here, he works with formulas built from equations involving states of a coalgebra and shows that two states are bisimilar just in case they satisfy the same set of so-called rigid and observable formulas. It appears that these formulas correspond to certain coequations over one color, but a more systematic comparison of his work to the co-Birkhoff theorem would be useful. (Goldblatt also works with a finitary logic, which goes beyond our considerations here.)

Along similar lines, it would be useful to generalize the notion of behavioral covarieties in order to distinguish those covarieties definable by coequations over, say, $n$ colors from those covarieties that require coequations over $m > n$ colors. We are unaware of any results in this direction thus far.

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